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Further Towards a Triadic Calculus

Part 3

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In Parts 1 and 2 of this paper, I attempted to augment an initial formulation of a calculus for triadas by Warren McCulloch and Roberto Moreno-Diaz, published as Quarterly Progress Report No. 84, from the Research Laboratory of Electronics, Massachusetts Institute of Technology, January 15, 1967, pp. 335-346. Definitions and operations (rotations, reflections, and shifts) were followed by binary operations (nonrelative products and sums) and by triadic operations (relative products and sums). We had begun to look at some unary and binary, or nonrelative, theorems. We now turn to triadic, or relative, theorems. [E1]-[E47] and (S1)-(S60) refer respectively to equations and sentences cited in Parts 1 and 2 of this paper.

Triadic, or Relative, Theorems

Let Qijk be the triada that results from the operation Δ(ABC), that is,

$$Q = Q_{ijk} = \sum_{123} A_{i23} \bullet B_{1j3} \bullet C_{12k} \qquad [E48]$$

Rotation of Q_{ijk} gives

$$\widehat{Q} = Q' = \widehat{Q_{ijk}} = Q'_{kij} \sum_{1'2'3'} X_{k2'3'} \bullet Y_{1'i3'} \bullet Z_{1'2'j} \qquad [E49]$$

To maintain synonymy, we note that A and i, B and j, and C and k are closely related. We can thus only hope to achieve synonymy if X and C, Y and A, and Z and B are put into some sort of equivalence. This equivalence has to do only with the external links to i, j, and k; we shall have to check, as well, on the internal links. We therefore set X = C', Y = A', and Z = B'.

$$\widehat{Q} = Q' = \widehat{Q_{ijk}} = \sum_{1'2'3'} C'_{k2'3'} \bullet A'_{1'i3'} \bullet B'_{1'2'j} \qquad [E50]$$

To remind us that the internal numbering sequence is arbitrary, we have used primed numbers. Thus, l' must be the same l' within Q', but there is no reason to expect l' in Q' to be the same as 1 in Q. Thus, the internal structure is the same. That is to say, the colligative term shared by A and B (or A' and B') is 3 (or l'); the colligative term shared by A or C (or A' or C') is 2 (or 3'); the colligative term shared by B or C (or B'or C') is I (or 2'). But note that if we rotate C, A, and B we may construct a

$$Q'' = \underset{\cdot}{\Delta}(\widehat{C}\widehat{A}\widehat{B}) \qquad [E51]$$

$$Q'' = Q''_{kij} \sum_{123} C''_{k23} \bullet A''_{1i3} \bullet B''_{12j} \qquad [E52]$$

with which compare

$$Q'_{kij} = \sum_{1'2'3'} C'_{k2'3'} \bullet A'_{1'i3'} \bullet B'_{1'2'j} \qquad [E53]$$

which corresponds exactly. We therefore conclude that we may remove the prime signs and write:

$$\widehat{Q_{ijk}} = Q_{ijk} \sum_{123} C_{k23} \bullet A_{1i3} \bullet B_{12j} = \underset{\cdot}{\Delta}(\widehat{C}\widehat{A}\widehat{B}), \text{ or if} \qquad [E54]$$

$$Q = \underset{\cdot}{\Delta}(ABC), \text{ then } \widehat{Q} = \widehat{\underset{\cdot}{\Delta}(ABC)} = \underset{\cdot}{\Delta}(\widehat{A}\widehat{B}\widehat{C}) \qquad [E55]$$

These operations are shown graphically in Fig. 37.

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The graphical solution, Fig. 37, indicates perhaps more visibly the rotation of a delta-product. To persuade ourselves of the maintenance of a synonymy, let

A = __gives__to__, (S61)

B = _leaves__for__, (S62)

and C = __tells on__to__ (S63)

$\overline{\underset{\cdot}{\Delta}(ABC)}$ then reads:

<u>i</u> gives 2 to 3, and 1 leaves <u>j</u> for 3, and 1 tells on 2 to <u>k</u> (S64)

Rotation, or $\widehat{\underset{\cdot}{\Delta}(ABC)}$, then reads:

<u>k</u> is told by 2' about 3', and to l' <u>i</u> gives 3', and for l', 2' leaves <u>j</u> (S65)

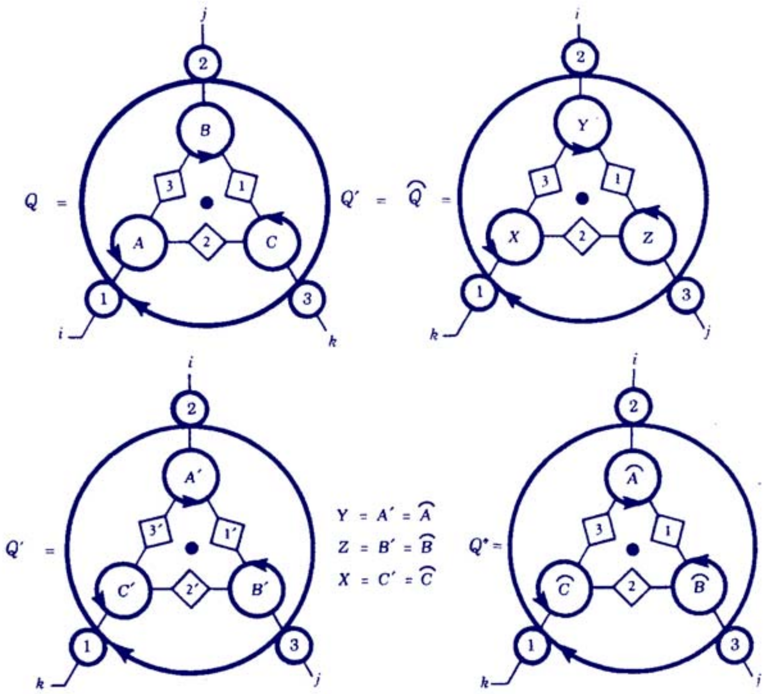

FIG. 37.

It is clear that the meanings are entirely equivalent, since the cardinal numbers in the above sentences refer to "some one," or "some other," or "yet some other" in any arbitrary sequence (here being simply the relative order in which they appear in the delta product.)

The reflection of Q_{ijk}, of

$$\breve{Q}_{ijk} = \overbrace{\sum_{123} A_{i23} \bullet B_{1j3} \bullet C_{12k}} \qquad \text{[E56]}$$

or

$$\breve{Q}_{ijk} = Q_{kji} \qquad \text{[E57]}$$

$$Q' = Q_{kij} = \sum_{1'2'3'} X_{k2'3'} \bullet Y_{1'i3'} \bullet Z_{1'2'j} \qquad \text{[E58]}$$

Again, to preserve synonymy, we assume that X is related to C, Y to B, and Z to A. (See Fig. 38.)

$$Q'_{kij} = \sum_{1'2'3'} C'_{k2'3'} \bullet B'_{1'j3'} \bullet A'_{1'2'i} \qquad \text{[E59]}$$

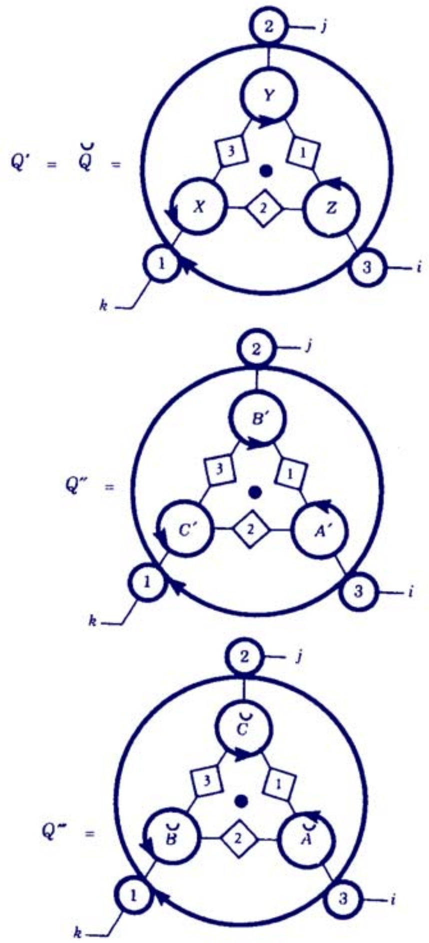

FIG. 38.

If we now reflect each of the triadas A, B, and C and use
$\overset{\smile}{A''} = A$, $\overset{\smile}{B''} = B$ and $\overset{\smile}{C''} = C$
we obtain the triada

$$Q'''_{kji} = \sum_{123} C''_{k23} \bullet B''_{1j3} \bullet A''_{12i} \qquad \text{[E60]}$$

We note the similarity of internal structure among the internal elements of Q" or Q" where Q''_{kji} has exactly the form of Q_{kji}; or,

$$Q''_{kji} = Q \qquad \text{[E61]}$$

The second element of C' (or of C") is the first element of B' (or of B"). The third element c C' (or of C") is the first element of A ; (or of A"). The third element of B' (or of B") is the second element of A; (or ofA"). At this stage, we can remove the prime signs from ot expressions for Q_{kji} to conclude that

$$\overset{\smile}{Q}_{ijk} = Q_{ijk} = \sum_{123} \overset{\smile}{C}_{12k} \bullet \overset{\smile}{B}_{1j3} \bullet \overset{\smile}{A}_{i23} \qquad \text{[E62]}$$

or if $Q = \underset{\cdot}{\Delta}(ABC)$, then

$$\overset{\smile}{Q} = \underset{\cdot}{\Delta}(\overset{\smile}{C}\ \overset{\smile}{B}\ \overset{\smile}{A}) \qquad \text{[E63]}$$

If T_{abc} is reflected, we obtain Fig. 39 where the origin is shifted from the first to the third element, and the direction of the arrow is reversed. For complex figures, every triada is reflected, as in Fig. 40.

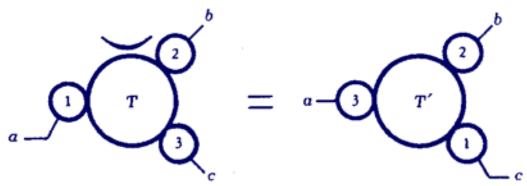

FIG. 39.

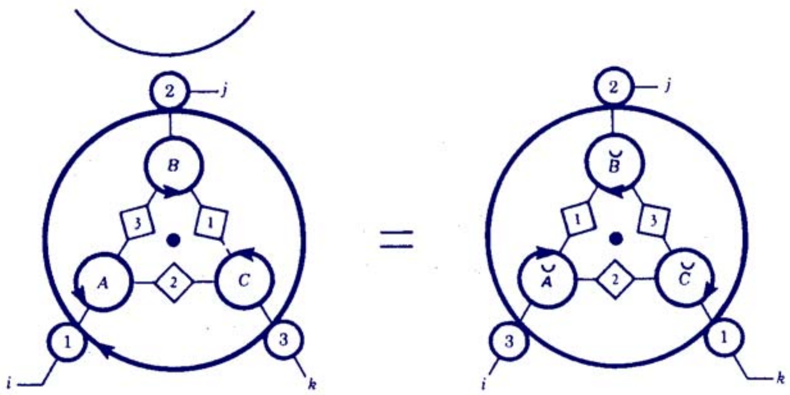

FIG. 40.

If A, B, and C are the triadas (61), (62), and (63), respectively, then $Q = \underset{\cdot}{\Delta}(ABC)$ reads, as before

i gives 2 to 3, and 1 leaves j for 3, and 1 tells on 2 to k (S66)

and $\overset{\smile}{Q} = \underset{\cdot}{\Delta}(\overset{\smile}{C}\ \overset{\smile}{B}\ \overset{\smile}{A})$ reads:

to k , 2' is told on by 3', and for l', j is left by 3', and to l', 2' is given by i (S67)

We note that the meanings of (S66) and S(67) are indeed synonymous.

By similar procedures, it is possible to show that

$$\overparen{\succ_{\bullet}\!-(A\ B\ C)} = {}_{\bullet}\!-\!\!\prec(\breve{C}\ \breve{B}\ \breve{A}) \qquad [E64]$$

The direct graphical solution is shown in Fig. 41.

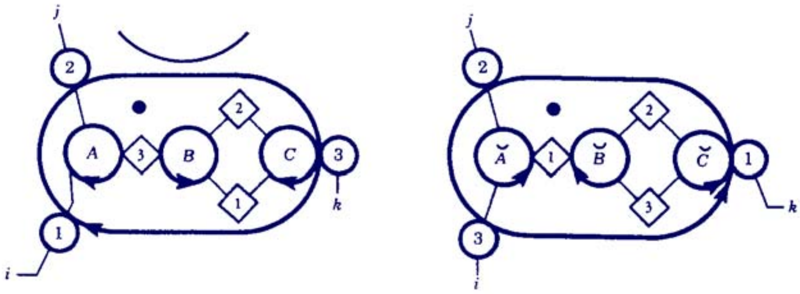

FIG. 41.

The figure for Q' is obtained simply by reflecting every triada of Q, satisfying the requirements of relative order consistency.

If Q is $\succ\!-(ABC)$ and Q' is $\breve{Q} = {}_{\bullet}\!-\!\!\prec(\breve{C}\ \breve{B}\ \breve{A})$ then Q:

__gives__to 3, and 1 leaves 2 for 3, and 1 tells on 2 to__ (S68)

Q′ :

to__, 2' is told on by 3', and for l', 2' is left by 3', and to l', __is given by__ (S69)

The direct graphical solution for

$$\overparen{{}_{\bullet}\!-\!\!\prec(A\ B\ C)} = \succ_{\bullet}\!-(\breve{C}\ \breve{B}\ \breve{A}) \qquad [E65]$$

is shown in Fig. 42, where again, Q' is obtained by reflecting every triada of Q, maintaining order consistency. Similarly, we can prove that

$$\overparen{\underset{+}{\Delta}(ABC)} = \underset{+}{\Delta}(\overset{\frown}{C}\,\overset{\frown}{A}\,\overset{\frown}{B}) \qquad [E66]$$

$$\overparen{\underset{+}{\Delta}(A\ B\ C)} = \underset{+}{\Delta}(\breve{C}\ \breve{B}\ \breve{A}) \qquad [E67]$$

$$\overparen{\succ_{+}\!-(A\ B\ C)} = {}_{+}\!-\!\!\prec(\breve{C}\ \breve{B}\ \breve{A}) \qquad [E68]$$

$$\overparen{{}_{+}\!-\!\!\prec(A\ B\ C)} = \succ_{+}\!-(\breve{C}\ \breve{B}\ \breve{A}) \qquad [E69]$$

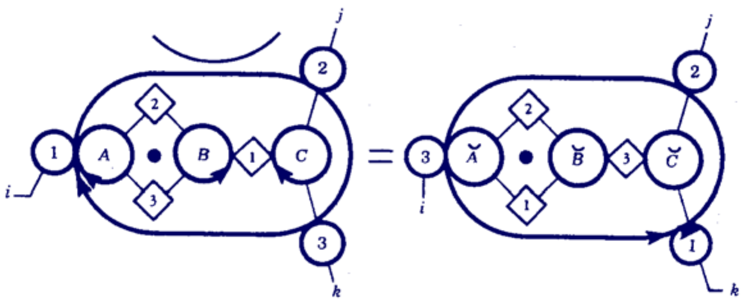

FIG. 42.

* *Constant Triadas*

We define five particular triadas that we shall use in the calculus.

a. ***Universal triada***, I_{ijk} , or simply I , is the triada

"__, __and__are individuals." (S70)

It has the following properties: Let A be any triada; then

$A + I = I$ and [E70]

$A \bullet I = A$ [E71]

It is clear that

$\breve{I} = I$ and [E72]

$\overset{\frown}{I} = I$ (see Fig. 43) [E73]

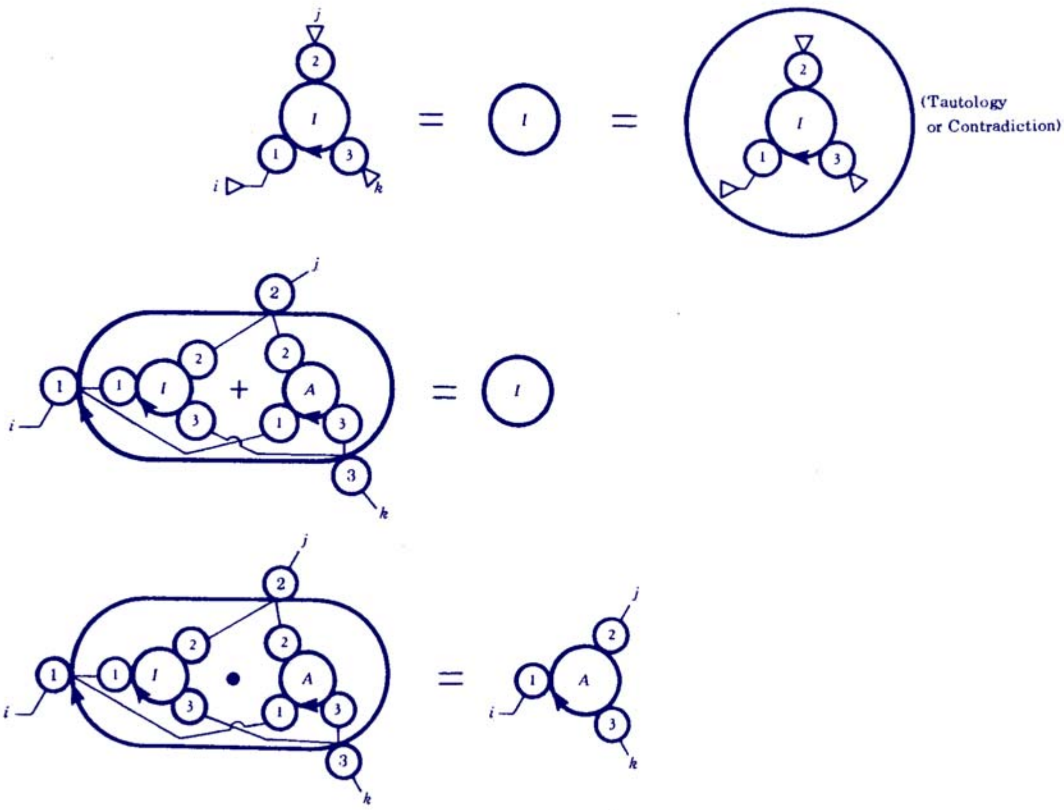

FIG. 43.

b. ***Null triada***, θ, or θ_{ijk}, is the triada

"neither__nor__nor__are individuals." (S71)

Let A be any triada; then

$A + \theta = A$ and [E74]

$A \bullet \theta = \theta$. Also, [E75]

$\overset{\frown}{\theta} = \theta$ and [E76]

$\breve{\theta} = \theta$ (see Fig. 44) [E77]

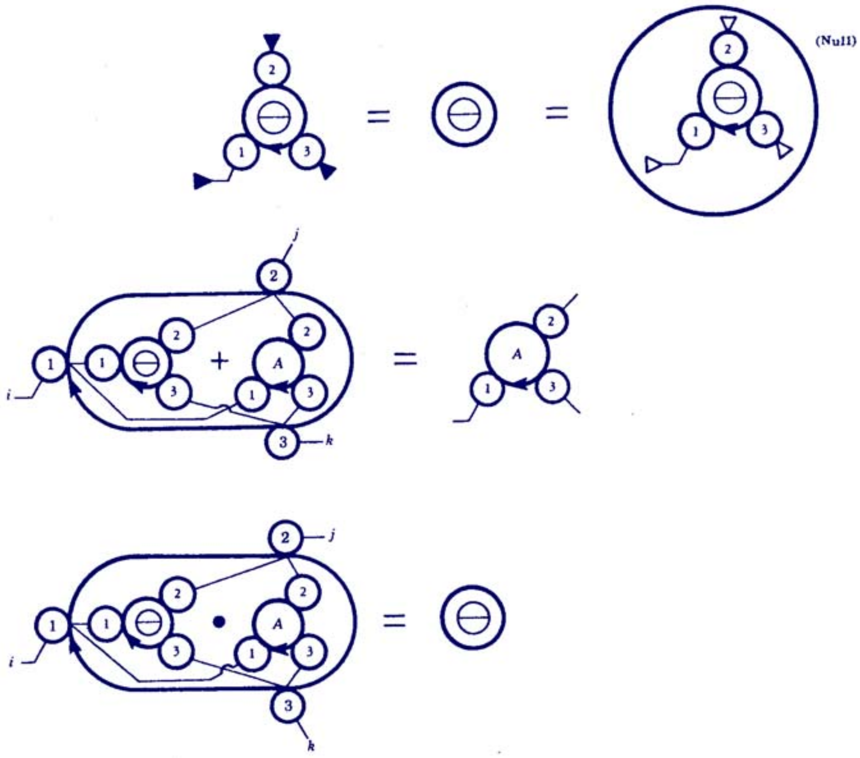

FIG. 44.

c. ***Left and Right Identities***, denoted by I_λ and I_ρ, respectively, are the following: I_λ is the triad

"__is an individual and__is identical to__ (S72)

I_ρ is the triada

"__is identical to__, and__is an individual" (S73)

It follows that

$$\breve{I}_\lambda = I_\rho \qquad [E78]$$

$$\breve{I}_\rho = I_\lambda \qquad [E79]$$

$$I_\lambda = \widehat{I_\rho} \qquad [E80]$$

(see Fig. 45.)

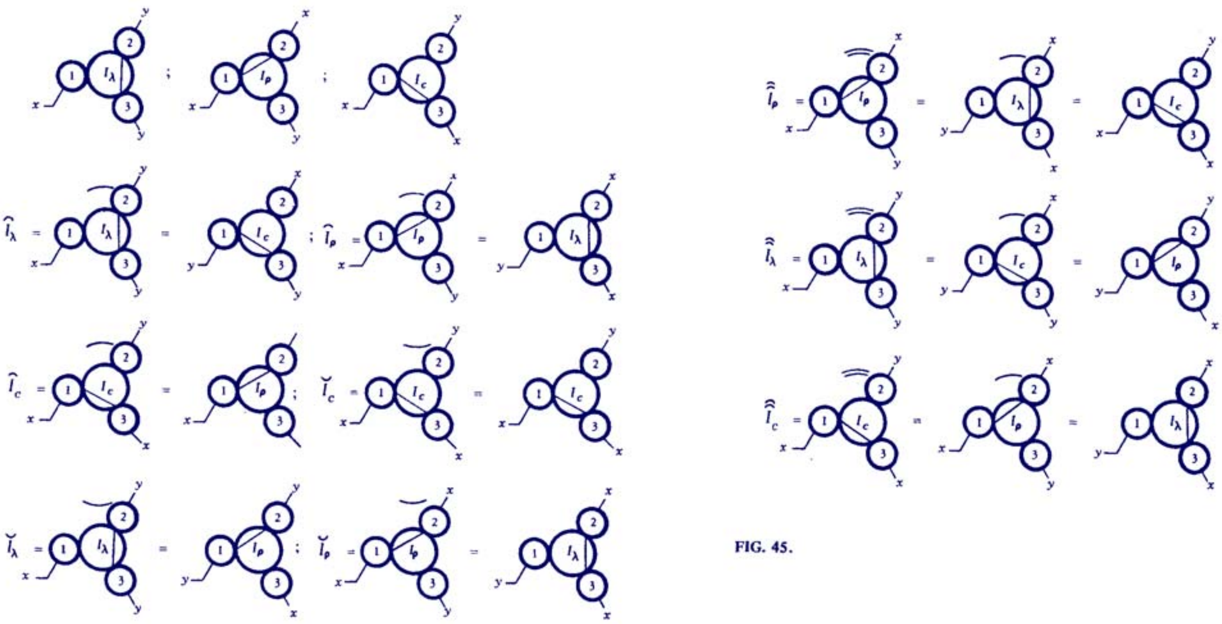

FIG. 45.

d. ***Central identity***, I_c is by definition,

$$I_c = \widehat{\widehat{I_\rho}} \qquad [E81]$$

or I_c is the triada,

"__is, while__is an individual, identical to__ (S74)

It follows that

$$\widehat{I_c} = I_\rho \quad \text{and} \qquad [E82]$$

$$\overset{\smile}{I_c} = I_c \quad \text{(see Fig. 46)} \qquad [E83]$$

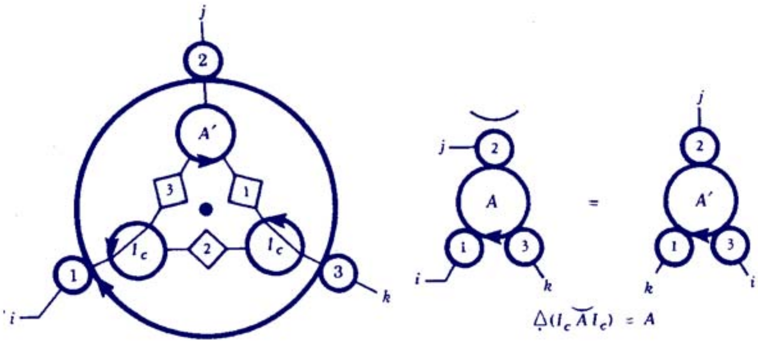

FIG. 46.

Let A be any triada; then

$$Q = \underset{\bullet}{\Delta}\,(I_c\,\overset{\smile}{A}\,I_c) = A \quad \text{(see Fig. 46)} \qquad [E84]$$

For example, let A be

"__gives__to__" (S75)

$Q = \underset{\bullet}{\Delta}\,(I_c\,\overset{\smile}{A}\,I_c) =$ then reads [E84]

"There are three individuals such that__, while 2 is an individual, is identical to 3; and 1 is given__by 3; and 1, while 2 is an individual, is identical to__" (S76)

end of *

Sentence 76 is the same as:

to__is given__by__ (S77)

Similarly,

$$R = \underset{\bullet}{\Delta}\,(I_\rho\, I_\rho\, \overset{\smile\smile}{A}) = \overset{\smile\smile}{A} \quad \text{(see Fig. 47)} \qquad [E85]$$

$$S = \underset{\bullet}{\Delta}\,(\overset{\frown}{\overset{\smile}{A}}\, I_\lambda\, I_\lambda) = \overset{\frown}{\overset{\smile}{A}} \quad \text{(see Fig. 48)} \qquad [E86]$$

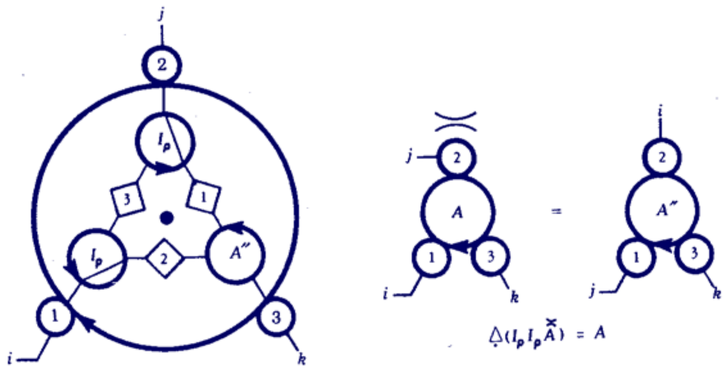

FIG. 47.

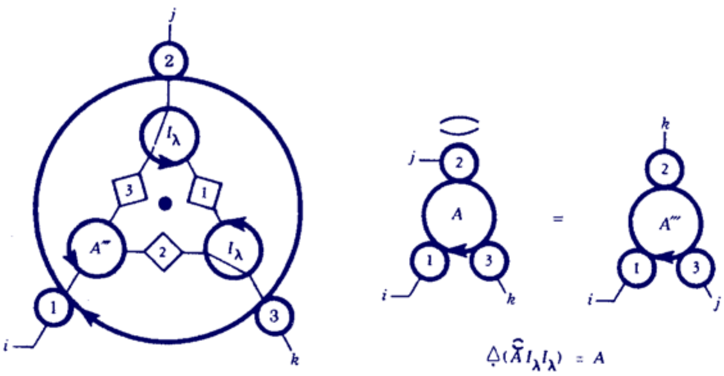

FIG. 48.

Table 2

$$I_c = \breve{I}_c = \overset{\frown}{I}_\lambda = \overset{\frown}{\overset{\frown}{I}}_\rho \qquad [E87]$$

$$I_\lambda = \breve{I}_\rho = \overset{\frown}{I}_\rho = \overset{\frown}{\overset{\frown}{I}}_c \qquad [E88]$$

$$I_\rho = \breve{I}_\lambda = \overset{\frown}{I}_c = \overset{\frown}{\overset{\frown}{I}}_\lambda \qquad [E89]$$

Theorem $\breve{R} = \underset{\bullet}{\Delta}(I_c\, R\, I_c)$ [E90]

Proof

let $Q = R = \underset{\bullet}{\Delta}(I_c\, \breve{R}\, I_c)$ by[E84] [E91]

then $Q' = \breve{Q} = \breve{\breve{R}} = R$ by[E4] [E92]

$Q' = \overset{\frown}{\underset{\bullet}{\Delta}(I_c\, \breve{R}\, I_c)} = R = \underset{\bullet}{\Delta}(\breve{I}_c\, \breve{\breve{R}}\, \breve{I}_c)$ by [E63] [E93]

$\breve{I}_c = I_c$ by [E4] and $\breve{\breve{R}} = R$ by [E4] [E94]

$\breve{R} = \underset{\bullet}{\Delta}(I_c\, R\, I_c)$ [E95]

See Fig. 49 for graphical proof.

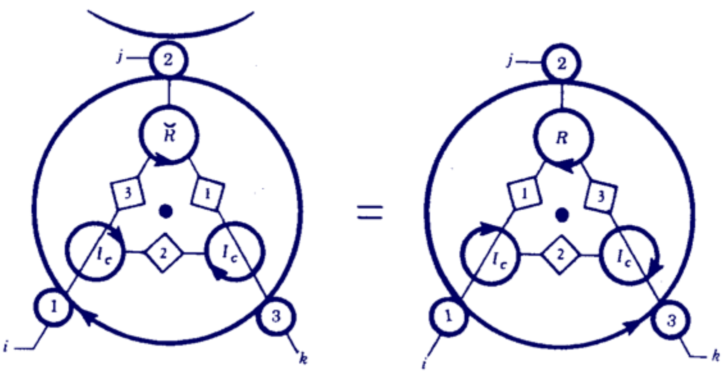

FIG. 49.

Let R = __gives__to__
(S78)

Then $Q = (I_c R I_c)$ reads

There are three individuals such that__,
while 2 is an individual, is identical to 3;

and 1 gives__to 3; and 1, while 2 is
an individual, is identical to__
(S79)

Sentence (S79) is equivalent to the reflection of (S78).

Theorem $$Q = \underset{\bullet}{\Delta}\,[\,\underset{\bullet}{\Delta}\,(A\,\overset{\frown}{\overset{\smile}{B}}\,I_\rho)\,I_\rho\,\overset{\smile}{\overset{\frown}{C}} = R = -\!\!\!\bullet\!\!<(ABC)$$ [E96]

Proof: (See Fig. 50)

let $$Q'_{i'j'k'} = \underset{\bullet}{\Delta}\,(A\,\overset{\smile}{\overset{\frown}{B}}\,I_\rho) = \sum_{1'2'3'} A_{i2'3'} \bullet \overset{\smile}{\overset{\frown}{B}}_{1'j3'} \bullet I_{\rho 1'=2'k}$$ [E97]

$$Q'' = \underset{\bullet}{\Delta}\,(Q'\,I_\rho\,\overset{\smile}{\overset{\frown}{C}})$$ [E98]

$$Q''_{i''j''k''} = \sum_{1''2''3''} Q_{i''2''3''} \bullet I_{\rho 1''=j''3} \bullet \overset{\smile}{\overset{\frown}{C}}_{1''2''k''}$$ [E99]

$$= \sum_{1''2''3''} \Big(\sum_{1'2'3'} A_{i'2'3'} \bullet \overset{\smile}{\overset{\frown}{B}}_{1'j'3'} \bullet I_{\rho 1'=2'k'}\Big)_{i''2''3''} \bullet I_{\rho 1''=j''3''} \bullet \overset{\smile}{\overset{\frown}{C}}_{1''2''k''}$$ [E100]

where i' = i", j' = 2", and k' = 3", or

$$Q''_{i''j''k''} = \sum_{1''2''3''} \Big(\sum_{1'2'3'} A_{i''2'3'} \bullet \overset{\smile}{\overset{\frown}{B}}_{1'2''3'} \bullet I_{\rho 1'=2'3''}\Big)_{i''2''3''} \bullet I_{\rho 1''=j''3''} \bullet \overset{\smile}{\overset{\frown}{C}}_{1''2''k''}$$ [E101]

$$-\!\!\!\bullet\!\!<(A\;B\;C) = \sum_{123} A_{i23} \bullet B_{123} \bullet C_{1jk})$$ [E102]

Thus $\overset{\smile}{\overset{\frown}{B}}_{2'2''3'} = B_{2''2'3'}$ and $\overset{\smile}{\overset{\frown}{C}}_{j''2''k''} = C_{2''j''k''}$ by [E37] [E103]

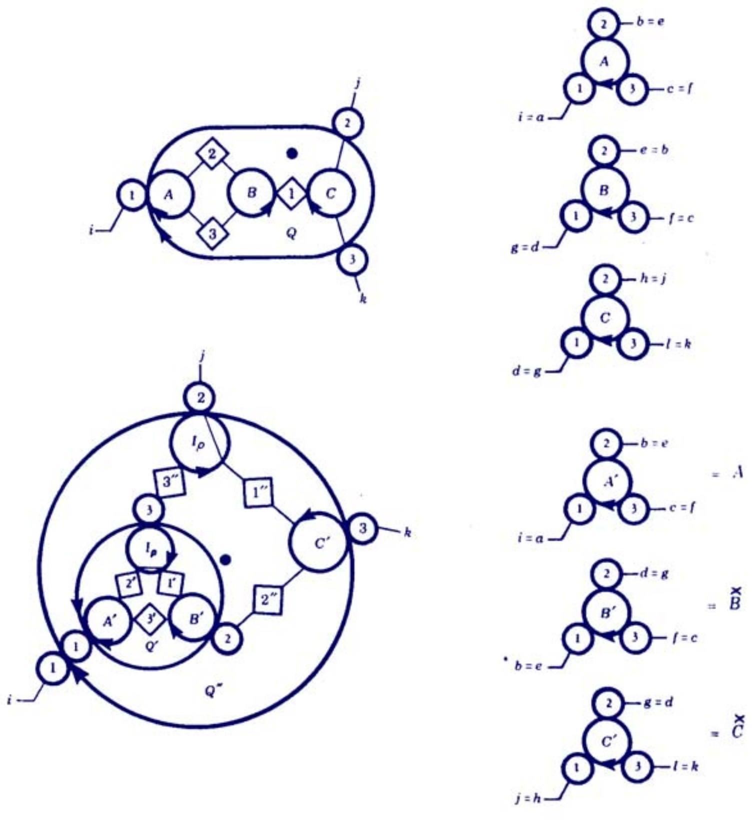

FIG. 50.

Now in either [E101] or [E102], the relative order of the arms and their connections are identical

The first of A is i,

The second of A is the second of B

The third of A is the third of B

The first of Bis the first of C

The second of B is the second of A

The third of B is the third of A

The first of C is the first of B

The second of C is j

and the third of C is k .

Thus one concludes that

$$\underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(A\,\overset{\frown}{\breve{B}}\,I_\rho)\;I_\rho\overset{\frown}{\breve{C}}] = \underset{\bullet}{-}\!\!\prec(ABC) \qquad [E104]$$

A meaning check for [E104]: Let A, B, C, be (S42), (S43), and (S44) as before.

Then $\underset{\bullet}{-}\!\!\prec(ABC)$ is:

__gives 2 to 3, and 1 leaves 2 for 3, and 1 tells on__to__
(S80)

$\underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(A\,\overset{\frown}{\breve{B}}\,I_\rho)\;I_\rho\overset{\frown}{\breve{C}}]$ is:

__gives 2' to 3' ; and l' is left by 2" for 3 ; and l' is identical to 2', while 3" is an individual; and 1" is identical to__, while 3' is an individual; and l' is told on by 2" to__
(S81)

Sentence (S81), though perhaps a tortuous sentence, is indeed a paraphrase of (S80). Because of the nature of the identity I_ρ, both graphs Q and Q" of Fig. 50 are the same. The introduction of the colligative terms 2' and 3" in the graph for Q' and Q° does not alter the meaning, for they are equivalent to saying only that

"someone is an individual"
(S82)

A $\underset{\cdot}{\Delta}$ product equivalent to $\succ\!\!\underset{\cdot}{-}$(ABC) is shown graphically as Fig. 51.

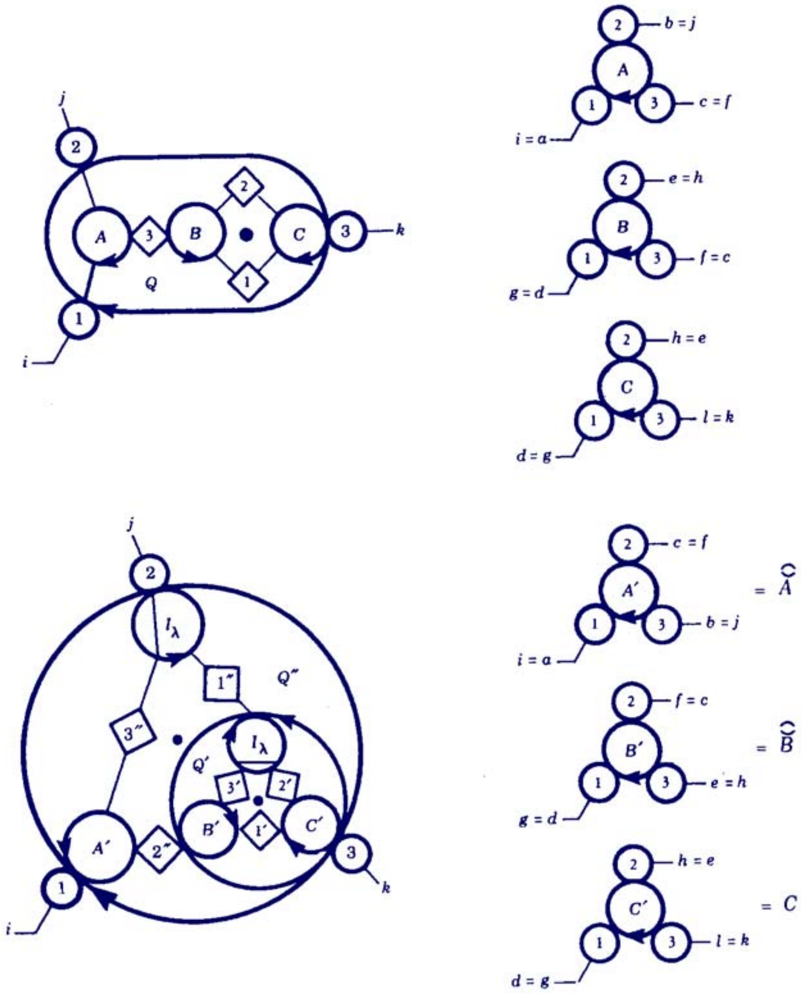

FIG. 51.

$$\underset{\cdot}{\succ\!-}(ABC) = \underset{\cdot}{\Delta}[\overset{\frown}{A}\, I_\lambda\, \underset{\cdot}{\Delta}(I_\lambda\, B\, \overset{\frown}{C})] \qquad [E105]$$

This has been constructed graphically simply by maintaining the consistency of interfac and internal connections.

Theorem $$\underset{\cdot}{\succ\!-}(ABC) = \underset{\cdot}{\Delta}[\overset{\frown}{A}\, I_\lambda\, \underset{\cdot}{\Delta}(I_\lambda\, B\, \overset{\frown}{C})] \qquad [E106]$$

By [E9] $$\underset{\cdot}{-\!\prec}(ABC) = \underset{\cdot}{\Delta}[\underset{\cdot}{\Delta}(A\, \overset{\asymp}{B}\, I_\rho)\, I_\rho \overset{\asymp}{C}] \qquad [E107]$$

Let $Q = \underset{\cdot}{-\!\prec}(RST)$ then $Q = \underset{\cdot}{\Delta}[\underset{\cdot}{\Delta}(R\, \overset{\asymp}{S}\, I_\rho)\, I_\rho \overset{\asymp}{T}]$ [E108]

$$\overset{\smile}{Q} = \underset{\cdot}{\succ\!-}(\overset{\smile}{T}\, \overset{\smile}{S}\, \overset{\smile}{R}) \quad \text{by [E65]} \qquad [E109]$$

$$\overset{\smile}{Q} = \underset{\cdot}{\Delta}[\overset{\asymp}{T}\, \overset{\smile}{I}_\rho\, \underset{\cdot}{\Delta}(R\, \overset{\asymp}{S}\, I_\rho)] \qquad [E110]$$

$$= \underset{\cdot}{\Delta}[\overset{\asymp}{T}\, \overset{\smile}{I}_\rho\, \underset{\cdot}{\Delta}(\overset{\smile}{I}_\rho \overset{\asymp}{S}\, \overset{\smile}{R})] = \underset{\cdot}{\Delta}[\overset{\frown}{T}\, I_\lambda\, \underset{\cdot}{\Delta}(I_\lambda \overset{\frown}{S}\, \overset{\frown}{R})] \qquad [E111]$$

$\overset{\smile}{T} = A$ then $T = \overset{\smile}{A}$ and $\overset{\frown}{T} = \overset{\frown}{A}$ [E112]

$\overset{\smile}{S} = B$ then $S = \overset{\smile}{B}$ and $\overset{\frown}{S} = \overset{\frown}{B}$ [E113]

$$\overset{\smile}{R} = C \qquad [E114]$$

$$\underset{\bullet}{\succ\!-}\,(ABC) = \underset{\bullet}{\Delta}\,[\overset{\circ}{A}\, I_\lambda\, \underset{\bullet}{\Delta}(I_\lambda\, \overset{\circ}{B}\, C)] \qquad [E115]$$

Note that [E107] is identical with [E97], arrived at graphically.

Using A, B, C sentences /S42), (S43), and (S44), the meaning of $\underset{\bullet}{\succ\!-}(ABC)$ is

__gives__to 1, and 1 leaves 2 for 3, and 1 tells on 2 to__
(S83)

$\underset{\bullet}{\Delta}\,[\overset{\circ}{A}\, I_\lambda\, \underset{\bullet}{\Delta}(I_\lambda\, \overset{\circ}{B}\, C)]$ is then:

__gives to 2 the gift 3; and while 1 is an individual, __is identical to 3;
and while 1 is an individual, 2' is identical to 3'; and l', leaves for 2, leaving 3';
and l' tells on 2' to__
(S84)

At this point, we might note that we have the equipment to determine at least some equivalent forms for the rotation of $\succ$ and $\prec$ products. Figure 52 illustrates a graphic solution to the _

Theorem : $$\underset{\bullet}{\succ\!-}\,\overset{\frown}{(ABC)} = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(\overset{\frown}{C}\, I_c\, \overset{\asymp}{B})\, \overset{\asymp}{A}\, I_c] \qquad [E116]$$

Proof:

If $$\underset{\bullet}{\succ\!-}\,(ABC) = \underset{\bullet}{\Delta}\,[\overset{\circ}{A}\, I_\lambda\, \underset{\bullet}{\Delta}\,(I_\lambda \overset{\circ}{B}\, C)\,] = Q \quad \text{by [E107]} \qquad [E117]$$

$$Q' = \underset{\bullet}{\Delta}\,(I_\lambda\, \overset{\circ}{B}\, C) \qquad [E118]$$

$$\underset{\bullet}{\succ\!-}\,\overset{\frown}{(ABC)} = \overset{\frown}{Q} = \underset{\bullet}{\Delta}\,\overset{\frown}{[A\, \overset{\circ}{I}_\lambda Q']} = \underset{\bullet}{\Delta}\,[\overset{\frown}{Q}{}'\, \overset{\frown}{\overset{\circ}{A}}\, \overset{\frown}{I}_\lambda] \qquad [E119]$$

$$\overset{\frown}{Q}{}' = \underset{\bullet}{\Delta}\,\overset{\frown}{(I_\lambda \overset{\circ}{B}\, C)} = \underset{\bullet}{\Delta}\,(\overset{\frown}{C}\, \overset{\frown}{I}_\lambda\, \overset{\frown}{\overset{\circ}{B}}) \quad \text{by [E55]} \qquad [E120]$$

or $$\overset{\frown}{Q} = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(\overset{\frown}{C}\, \overset{\frown}{I}_\lambda\, \overset{\frown}{\overset{\circ}{B}})\, \overset{\frown}{\overset{\circ}{A}}\, \overset{\frown}{I}_\lambda] \quad \text{by substituting [E125] in [E124]} \qquad [E121]$$

$$\overset{\frown}{I}_\lambda = I_c \quad \text{by [E88], or} \qquad [E122]$$

$$\overset{\frown}{Q} = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(\overset{\frown}{C}\, I_c \overset{\frown}{\overset{\circ}{B}})\, \overset{\frown}{\overset{\circ}{A}}\, I_c] = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}\,(\overset{\frown}{C}\, I_c\, \overset{\asymp}{B})\, \overset{\asymp}{A}\, I_c] \qquad [E123]$$

Figure 52 illustrates this proof.

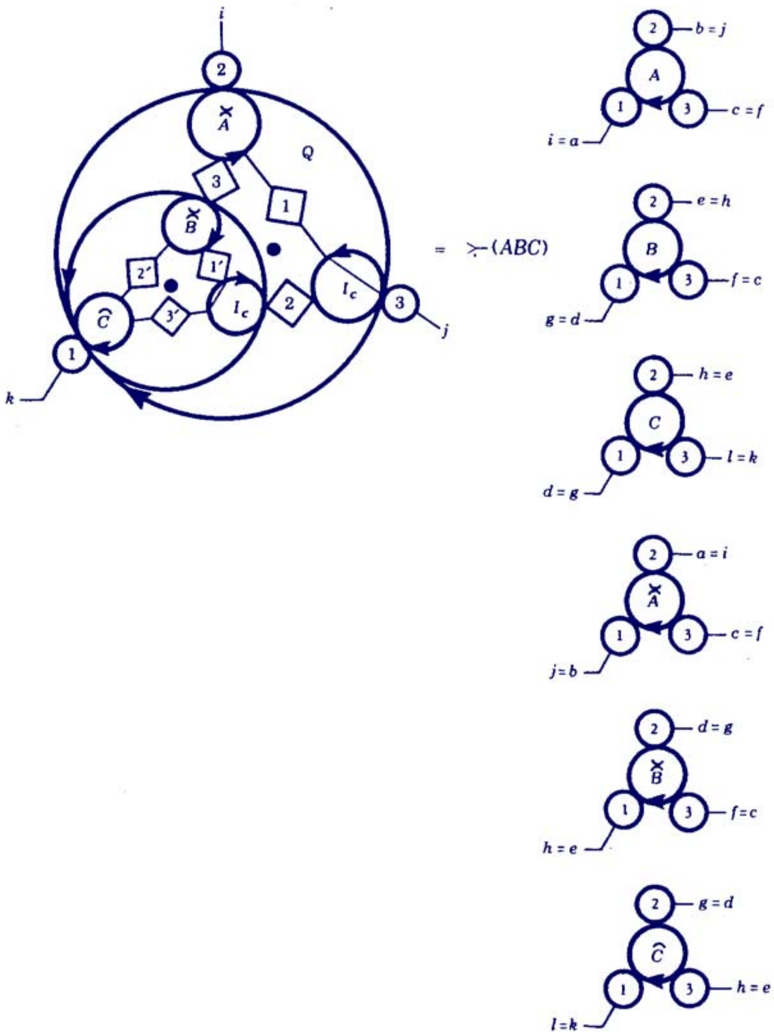

FIG. 52.

Using S61-63 for A, B, C, equation [E123] is illustrated by the sentence:

to__, 2'tells on 3; and l', while 2 is an individual,
is identical to 3', and for l', 2' leaves 3;
and 1 is given by__ to 3; and 1,
while 2 is an individual, is identical to__ (S85)

Similarly, we may prove the theorem, illustrated in Fig. 53,

Theorem : $$\underset{\bullet}{\prec}\widehat{(ABC)} = \underset{\bullet}{\Delta}\,[C\,\underset{\bullet}{\Delta}\,(I_\lambda \widehat{A}\,\breve{B})\,I_\lambda]$$ [E124]

Proof:

$$Q = \underset{\bullet}{\prec}(ABC) = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}(A\,\breve{\widehat{B}}\,I_\rho)\,I_\rho\,\breve{\widehat{C}}] \quad \text{by [E97]}$$ [E125]

$$Q = \underset{\bullet}{\Delta}\,[Q'\,I_\rho\breve{\widehat{C}}] \quad \text{where} \quad Q' = \underset{\bullet}{\Delta}\,(A\,\breve{\widehat{B}}\,I_\rho)$$ [E126]

$$\widehat{Q} = \underset{\bullet}{\Delta}\,[\widehat{\breve{\widehat{C}}}\,\widehat{Q'}\,\widehat{I_\rho}] \quad \text{and}$$ [E127]

$$\widehat{Q'} = \underset{\bullet}{\Delta}\,(\widehat{I_\rho}\widehat{A}\,\widehat{\breve{\widehat{B}}}) \quad \text{by [E55]}$$ [E128]

$$\widehat{I_\rho} = I_\lambda \quad \text{by [E89]; and} \quad \widehat{\breve{\widehat{C}}} = \breve{C} \quad \text{by [E40]}$$ [E129]

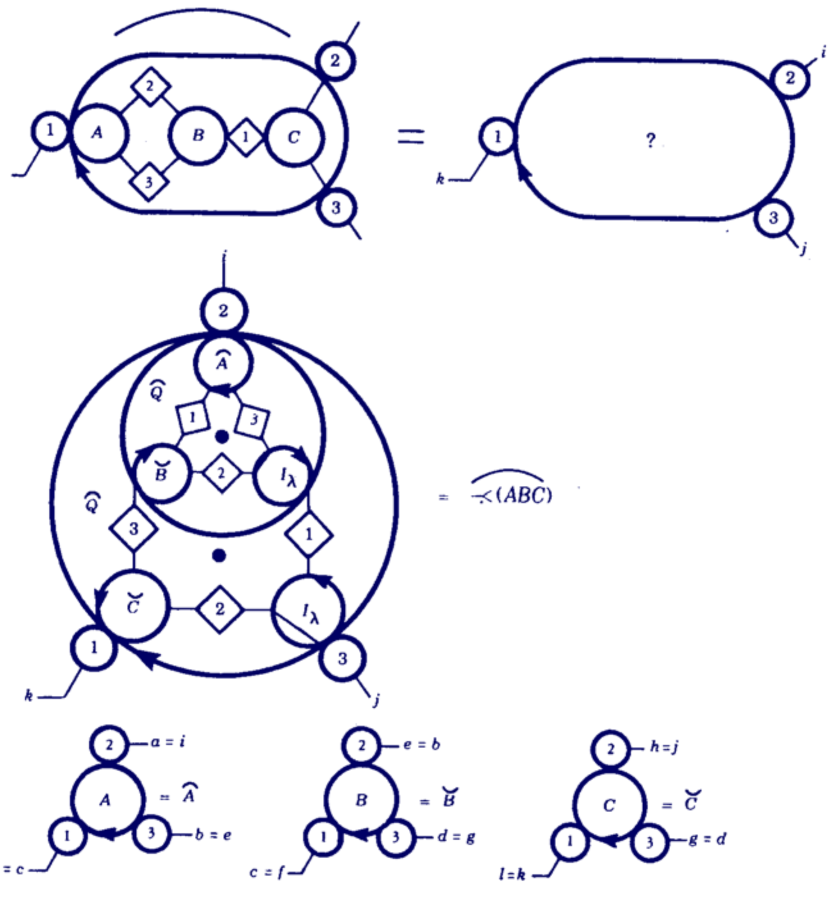

FIG. 53.

Thus $$\widehat{Q} = \underset{\bullet}{\prec}\widehat{(ABC)} = \underset{\bullet}{\Delta}\,[\breve{C}\,\underset{\bullet}{\Delta}(I_\lambda\widehat{A}\,\breve{B})\,I_\lambda\,]$$ [E130]

To assure ourselves that a rotation of a $\succ$ product may be represented by such equations as E116, we shall rotate the $\succ$ product twice again; we expect to find that the result is what we begin with, for in general,

$$\widehat{\widehat{\widehat{Q}}} = Q \quad \text{by [E7]}$$ [E131]

These operations are shown graphically in Fig. 54 and Fig. 55. Compare also Fig. 51.

Let $Q = \succ\!\!\cdot\,(ABC)$. Then by E[121],
[E132]

$$Q' = Q = \overset{\frown}{\succ\!\!\cdot\,(ABC)} = \underset{\bullet}{\Delta}\,[\underset{\bullet}{\Delta}(\overset{\frown}{C}\, I_c \breve{\overset{\frown}{B}})\, \breve{\overset{\frown}{A}}\, I_c] = \underset{\bullet}{\Delta}(RST)$$ [E133]

where $R = \underset{\bullet}{\Delta}(\overset{\frown}{C}\, I_c \breve{\overset{\frown}{B}})$ [E134]

$S = \breve{\overset{\frown}{A}}$ and [E135]

$T = I_c$ [E136]

$$Q'' = \overset{\frown}{\overset{\frown}{Q}} = \overset{\frown}{\underset{\bullet}{\Delta}(RST)} = \underset{\bullet}{\Delta}\,[\overset{\frown}{T}\,\overset{\frown}{R}\,\overset{\frown}{S}]$$ by[55] [E137]

$$Q'' = \underset{\bullet}{\Delta}[\overset{\frown}{I_c}\, \overset{\frown}{\underset{\bullet}{\Delta}(\overset{\frown}{C}\, I_c\, \overset{\frown}{B})}\, \overset{\smile}{\overset{\frown}{A}}]$$ [E138]

$$= \underset{\bullet}{\Delta}[\overset{\frown}{I_c}\, \underset{\bullet}{\Delta}(\overset{\smile}{\overset{\frown}{B}}\, \overset{\frown}{\overset{\frown}{C}}\, \overset{\frown}{I_c})\, \overset{\smile}{\overset{\frown}{A}}]$$ by[55] [E139]

$\overset{\frown}{I_c} = I_\rho$ by [E90]; and $\overset{\smile}{\overset{\frown}{B}} = \breve{B}$ and $\overset{\smile}{\overset{\frown}{A}} = \breve{A}$ by [E39]
[E140]

Thus $$\overset{\frown}{\overset{\frown}{\succ\!\!\cdot\,(ABC)}} = Q'' = \underset{\bullet}{\Delta}\,[I_\rho\, \underset{\bullet}{\Delta}(\breve{B}\, \overset{\frown}{\overset{\frown}{C}}\, I_\rho)\, \breve{A}]$$ [E141]

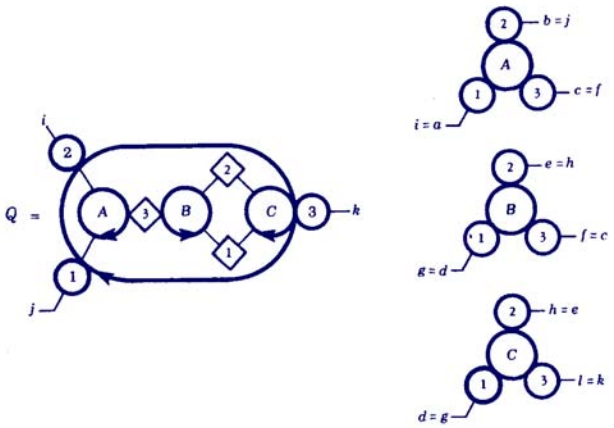

FIG. 54.

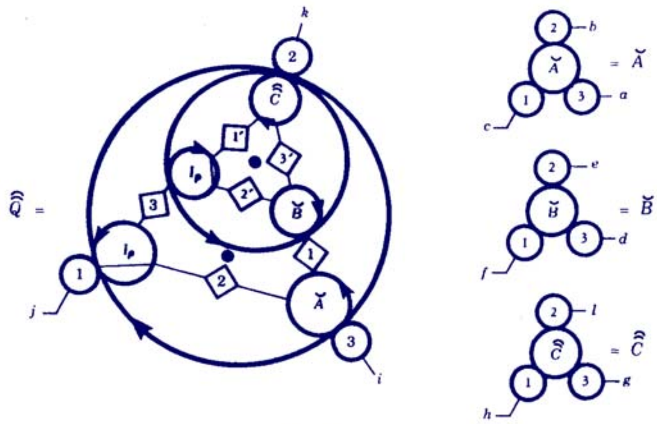

FIG. 55.

Now let $Q''' = \overset{\frown}{Q''} = \overset{\frown}{\underset{\bullet}{\Delta}\,[R'\, S'\, T']}$ where [E142]

$R' = I_\rho$ [E143]

$S' = \underset{\bullet}{\Delta}\,(\breve{B}\, \overset{\frown}{\overset{\frown}{C}}\, I_\rho)$ [E144]

$T' = \breve{A}$ [E145]

$$Q''' = \underset{\bullet}{\Delta}(\overset{\frown}{T}'\ \overset{\frown}{R}'\ \overset{\frown}{S}')$$ by [E55] [E146]

$$\overset{\frown}{T}' = \overset{\frown}{\underset{}{A}}$$ [E147]

$$\overset{\frown}{R}' = \overset{\frown}{I_\rho} = I_\lambda$$ [E148]

$$\overset{\frown}{S}' = \underset{\bullet}{\Delta}(\overset{\frown}{I_\rho}\ \overset{\frown\frown}{B}\ \overset{\frown}{C}) = \underset{\bullet}{\Delta}(I_\lambda\ \overset{\frown}{B}\ C)$$ [E149]

$$Q''' = \overset{\frown\frown\frown}{\succ\!\!\cdot\,(ABC)} = \underset{\bullet}{\Delta}[\overset{\frown}{A}\ I_\lambda\ \underset{\bullet}{\Delta}(I_\lambda \overset{\frown}{B}\ C\ )]$$ which equals [E121] [E150]

Similarly, for the $\prec$ product derivations, we check by rotating a $\prec$ product three times.

$$Q = \cdot\!\!\prec(ABC) = \underset{\bullet}{\Delta}[\underset{\bullet}{\Delta}(A\ \overset{\smile\smile}{B}\ I_\rho)\ I_\rho\ \overset{\smile\smile}{C}]$$ by [E91] [E151]

$$\overset{\frown}{Q} = \overset{\frown}{\cdot\!\!\prec(ABC)} = Q' = \underset{\bullet}{\Delta}[\overset{\smile}{C}\ \underset{\bullet}{\Delta}(I_\lambda \overset{\frown}{A}\ \overset{\smile}{B})\ I_\lambda]$$ by [135] [E152]

$$Q'' = \overset{\frown\frown}{Q} = \overset{\frown}{\underset{\bullet}{\Delta}[R\ S\ T]}$$ where $R = \overset{\smile}{C}$ and $S = \underset{\bullet}{\Delta}(I_\lambda \overset{\frown}{A}\ \overset{\smile}{B})$ and $T = I_\lambda$ [E153]

$$Q'' = \underset{\bullet}{\Delta}[\overset{\frown}{R}\ \overset{\frown}{S}\ \overset{\frown}{T}] = \underset{\bullet}{\Delta}[\overset{\frown}{I_\lambda} \overset{\frown\smile}{C}\ \Delta(\overset{\frown\frown}{B}\ \overset{\frown}{I_\lambda} \overset{\frown\frown}{A})]$$ by [55] [E154]

$$\overset{\frown}{I_\lambda} = I_c$$ by [88] [E155]

$$Q'' = \underset{\bullet}{\Delta}[I_c \overset{\frown\smile}{C}\ \underset{\bullet}{\Delta}(\overset{\frown\frown}{B}\ I_c \overset{\frown\frown}{A})]$$ [E156]

$$Q''' = \overset{\frown}{Q}'' = \overset{\frown\frown\frown}{Q} = Q = \cdot\!\!\prec(ABC) = \overset{\frown}{\underset{\bullet}{\Delta}[U\ V\ W]}$$ where [E157]

$$U = I_c \text{ and } V = \overset{\frown\smile}{C} \text{ and } W = \underset{\bullet}{\Delta}(\overset{\frown\frown}{B}\ I_c \overset{\frown\frown}{A})$$

$$Q''' = \underset{\bullet}{\Delta}[\overset{\frown}{U}\ \overset{\frown}{V}\ \overset{\frown}{W}]$$ by [88] [E158]

$$Q''' = \underset{\bullet}{\Delta}[\underset{\bullet}{\Delta}(\overset{\frown\frown\frown}{A}\ \overset{\frown\frown\smile}{B}\ \overset{\frown}{I_c})\ \overset{\frown}{I_c}\ \overset{\frown\frown\smile}{C}]$$ [E159]

$$\overset{\frown}{I_c} = I_\rho \text{ by [E90] and } \overset{\frown\frown\smile}{B} = \overset{\smile\smile}{B} \text{ and } \overset{\frown\frown\smile}{C} = \overset{\smile\smile}{C} \text{ and } \overset{\frown\frown\frown}{A} = A$$ [E160]

$$Q''' = Q = \underset{\bullet}{\Delta}[\underset{\bullet}{\Delta}(A\ \overset{\smile\smile}{B}\ I_\rho)\ I_\rho\ \overset{\smile\smile}{C}] = \text{ [E91]}$$ [E161]

Clearly, equivalent $\succ$ and $\prec$ sums may be derived following exactly the same procedure

Computer-aided Triadic Logic Development

Let us look a little more closely at our notation and especially at the graphical techniques we have used during the course of developing our triadic logic.

We might begin by noting Jerome Bruner's three modes of processing information – the enactive, the iconic, and the symbolic. (*Towards a Theory of Instruction*, Norton (New York, 1968)). To acquire the meanings of 'rotate' and 'reflect', it seems helpful to cut six approximately equilateral triangles out of cardboard, several inches on a side, labeling the apices a, b, and c. (See Fig. 56.) One apex is marked as the "origin" (a hole is punched, or the apex notched, or any other obvious mark visible on both sides will do). Which apex receives the origin mark and which letter goes on what apex is determined from Table 1, Fig. 33. One side (arbitrarily called the "top") is placed with the origin-marked apex leftward, the second apex upward, and the third apex rightward. In a clockwise direction, then, A_{abc} would be represented on the cardboard model by 'a' on the origin, leftward apex, 'b' on the upper apex, and 'c' on the right-hand apex. To complete the model, "A_{abc}" and an example as "a gives b to c" are added to the cardboard triangle. For the same of symbolic consistency, a circle with its origin and "arms" may be drawn in the center of the triangle as shown by Fig. 56, so that the completed figure looks like the upper left figure of Fig. 33, with the arms terminating in the apices of the model.

The reverse side of the model is then drawn in, using the "top" side to determine which apex is which. For the "bottom" side, of course, the order will be counter-clockwise, though the name (A_{abc}) and the example will be the same.

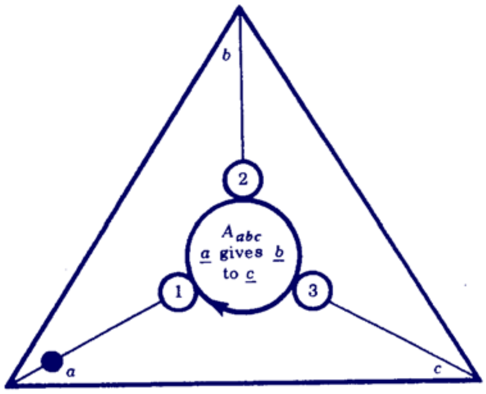

FIG. 56. Top side.

To give a more immediate ("enactive") sense of rotation and reflection, a circle may be cut out of such a figure. The circle should include its origin mark. That circle corresponds to the "transistor" of Fig. 6c. The connections to the arms, if they are placed 120^0 apart on the circumference of the circle, will permit a "socket" consisting of three arms labeled a, b, and c, respectively to match the circle when placed under the movable circle. By actually carrying out the unary operations physically on the model, rotation and reflection may be made meaningful. By copying the results onto the six cardboard triangles, a sense of relevance to the diagram of Fig. 33 and the example of Table 1 may be established. These cardboard pictoral ("iconic") representations then are helpful in working out more elaborate diagrams and the more abstract ("symbolic") notation.

Note that complex triadas (Δ, $\succ\!-$, and $-\!\prec$ sums and products) may be treated similarly, if necessary.

In the working out of proofs using the graphical representation, it becomes obvious that much of the labor involved reduces to rather trivial, mechanical operations (such as describing a particular triad in terms of its rotations) and/or reflection from any form). Much of this kind of labor should lend itself to interactive graphical computer systems. A set of programs designed to carry out such operations graphically as well as symbolically would permit a user to generate quickly and accurately an enormous number of lemmas and proofs, some of which might be interesting. We now examine more closely a particular proof with the aim of observing what kinds of interactions may be useful in such a system.

Let us observe the process of constructing a graphical proof. For our example, let us revert to Fig. 53, the graphical solution to [E129], which results in the peculiar $-\!\prec$ product equivalent to

$$\underset{\bullet}{\Delta}\,[\breve{C}\,\underset{\bullet}{\Delta}(I_{\lambda}\overset{\frown}{A}\,\breve{B})\,I_{\lambda}\,]$$

Let us begin by constructing the product which is to be rotated. We should expect to have some standard figures from which to begin assembling our complex triadas. Figure 57 illustrates some of these, to which we may add an alphameric set of capital and minuscule letters and the numbers 0-9, and the prime sign, and the Greek letters a and p, perhaps also Σ and Π.

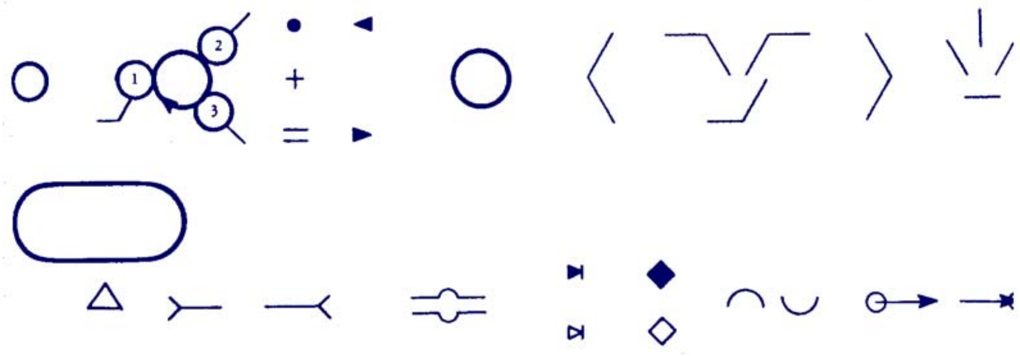

FIG. 57.

To construct a simple triada, say A_{abc}, we begin with the triada figure, add the appropriate minuscule letters a, b, c to the arms, and add capital letter A inside the figure. A is defined by the following schedule:

$$A = A_{abc}; \quad \hat{1}A = a; \quad \hat{2}A = b; \quad \hat{3}A = c$$

The circonflex sign is used here to indicate which arm is to be assigned what minuscule letter.

To construct a complex triada, we begin with three simple triadas A, B, and C, shift and arrange them to form the desired pattern. For example, the triada −<(ABC) could begin by bringing Aabc, Bdef, and Cghl reasonally linear order, as in Fig. 58.

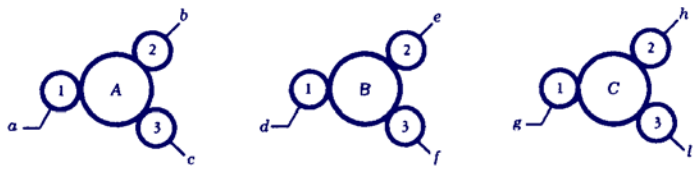

FIG. 58.

Our −< product is constructed by making a, h, and 1, external (unbound) members, and connecting all the others internally (bound members) according to the schedule shown in Table 3.

Such schedules, once defined, should certainly be part of any useful computer's repetory. One should be able to enter definitions into a system either graphically (i.e., by actually causing the desired connections to be made) or by some other means, such as a keyboard on which to type out the formal equivalences, such as those shown in Table 3.

TABLE 3.

$Q = -\!\!< (ABC)$

$\hat{1}Q = i;$	$d = \hat{1}B = 1$
$\hat{2}Q = j;$	$e = \hat{2}B = 2$
$\hat{3}Q = k;$	$f = \hat{3}B = 3$
$a = \hat{1}A = i;$	$g = \hat{1}C = 1$
$b = \hat{2}A = 2$	$h = \hat{2}C = j$
$c = \hat{3}A = 3$	$l = \hat{3}C = k$

Causing the graphical connections to be made results in the desired −< product, shown in Fig. 59.

Turning B around and shifting slightly the apices of B then results in a perhaps more esthetically pleasing, symmetrical diagram, as shown in Fig. 60, which also shows the triada moved a little closer together.

From Table 3, the internal conditions can be summarized by saying that

$$b = e = 2$$
$$c = d = 3$$
$$f = g = 1$$

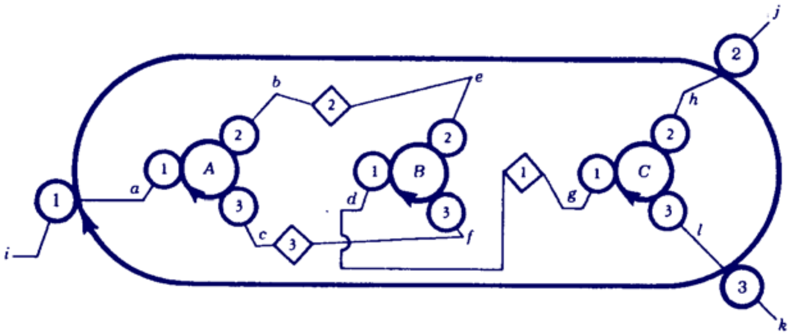

FIG. 59.

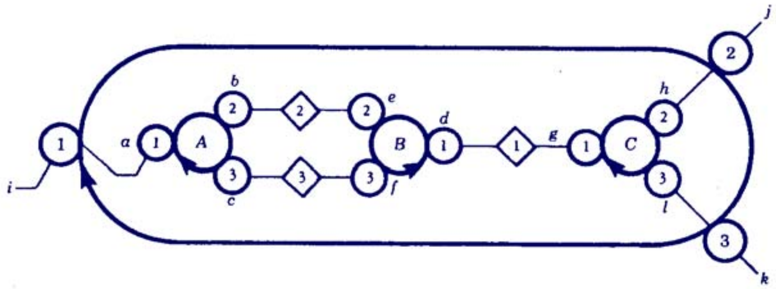

FIG. 60.

The external conditions can be summarized by saying that

$$a = i$$
$$h = j$$
$$1 = k$$

If we now attempt to rotate Q of Fig. 60, we begin by finding something like Q' in Fig. 61, where we note at once that we have violated the principle of internal and interface consistency. For this reason, Fig. 61 is shown with Q' dotted, for the construction is not well-formed. If we now manipulate the triadas A, B, and C to try to restore the desired consistency, we find Fig. 62 satisfies our needs.

Where $$A' = A_{abc} = \overset{\smile}{A}$$

$$B' = B_{efd} = \overset{\frown\frown}{B}$$

$$C' = C_{lgh} = \overset{\frown}{C}$$

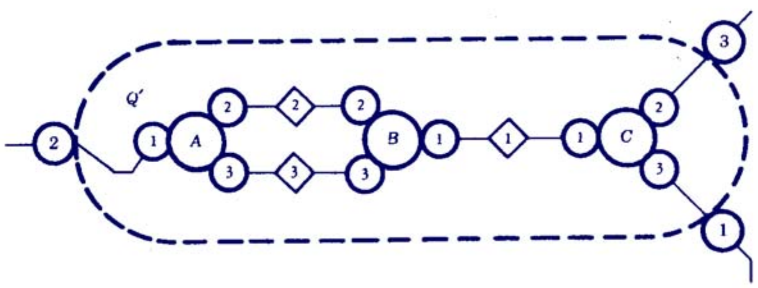

FIG. 61.

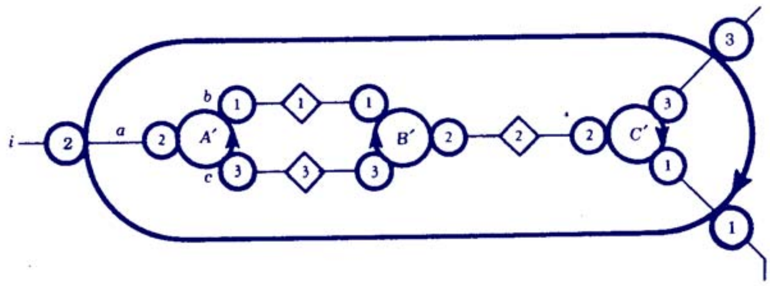

FIG. 62.

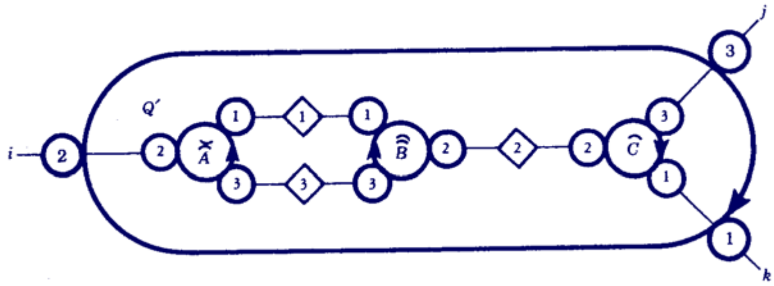

FIG. 63

Replacing these triadas in the figures, we find Q' in Fig. 63.

This construction permits us to fill in the mysterious blank of Fig. 53, for whatever late good may accrue from exploring this particular path.

Rotation of Q'gives us Q' or Q ; as shown in Fig. 64.

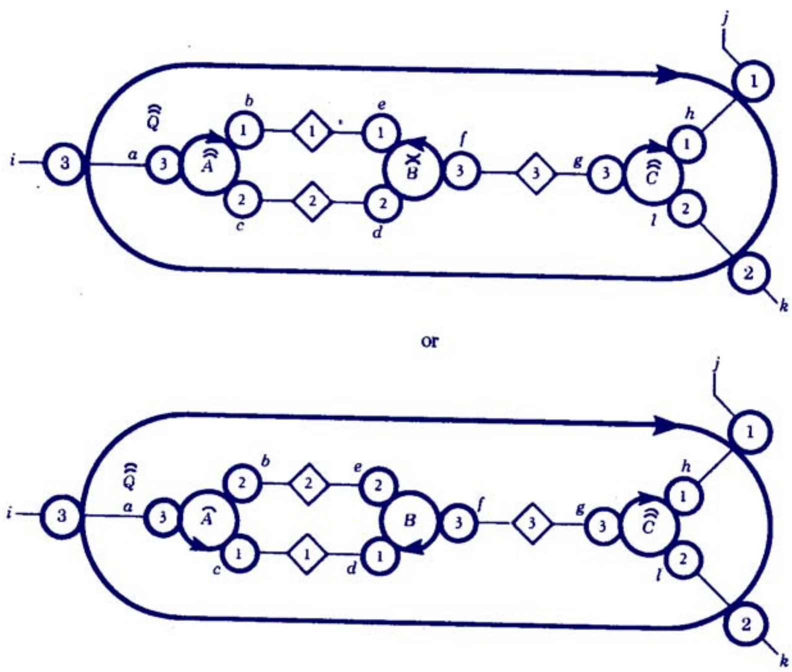

FIG. 64.

These are not unique solutions, for the pair of triadas A' and B' is constrained only by it; third arms, and the pairs communal first and second arms may be correctly connected in either of the two positions shown.

Rotation once more yields Q of Fig. 60 again, as expected, but also in Fig. 61.

From Figs. 60 and 65, then, we conclude that

$$\dot{-}\!\!<(ABC) \text{ and } \dot{-}\!\!<(\overset{\frown}{A}\,\overset{\frown}{B}\,C)$$

are equivalent.

A sentence for both: If A, B, C, are (S42), (S43), and (S44), then $\dot{-}\!\!<(ABC)$ is

__gives 2 to 3; and 1 leaves 2 for 3; and 1 tells on__to__
(S86)

and $\dot{-}\!\!<(\overset{\frown}{A}\,\overset{\frown}{B}\,C)$ is

__gives to 2, the gift 3; and 1 leaves for 2, leaving 3; and 1 tells on__to__
(S87)

We have perhaps toyed enough with the graphs that a summary of what we seem to nee for an interaction graphical system may be in order in this point. We note the following operations:

1. Construction of simple triadas, with arbitrary name and arbitrarily labeled apices, ordered arms indicated.
2. Physical turning, flopping over, and translation of triadas.
3. Physical bunching and spreading of apices along a circumference of a triada.
4. Stretching and placement of triadas to form `envelopes' around an arbitrary collection of triadas.
5. Connection and disconnection of apices to other apices.
6. Generation, placement and labeling of colligative terms quantifiers (◊ and ◆).
7. Rotation of triadas.
8. Reflection of triadas.

Further desirable features are:

9. Enlargement and shrinking of figures (perhaps in steps rather than continuously).
10. Automatic maintanance of formal equivalences (equations and subscripts).
11. Automatic generation of example sentences.
12. Automatic maintainance of "boundary conditions" and internal consistency.

Only Items 10-12 seem to require any comment, for Items 1-9 seem clearly within existing graphical system capability. Though some arbitrary and perhaps time-consuming decisions may need to be made (e.g., "no triada may approach the 'envelope' of another triada by less than X millimeters").

Automatic Maintainance of Formal Equivalences

Because of the traditions of logic if for no other reason, it seems desirable to be able to convert any graphic structure to an equivalent symbolic one.

We have noted already, for example, that we expect to have something like a reactive keyboard with which to communicate with the system. During the process of definition, for example, one way of equating symbolic notation with graphical manipulation is to introduce both and equate them deliberately. Thus, for example, "rotation" must first be defined as both a graphical operation (the conversion, for example, shown in Fig. 62) and a notational operation (the conversion, for example of A_{abc} to A_{cab}, or the conversion of A to $\widehat{A}$), and also a cross reference equating the two.

It should be clear that such convertibility requires greater computer capability than either symbolic conversion or graphical conversion alone. But because symbolic and graphical presentations are useful and desirable for different purposes and on different occasions, such conversion from one to the other should be available.

Automatic conversion from graph to symbol would provide a hardcopy summary of a proof determined graphically, for example. Furthermore, rather than attempting to store vast amounts of information about many possible different graphs, it seems more reasonable to storé the graphic information needed to begin graphical interaction, but by in large to store only symbolic representations and the general rules for converting these into graphical equivalents. The proportion of stored graphical data relative to rules for processing graphical data is a variable ratio, and its optimum will presumably change from time to time and must be determined experimentally. Factors which could affect the ratio include the hardware used, the available software, the speed of operation, the repetitive nature of particular figures, the processing efficiency of the general rules, the ease with which complex definitions can be built up from simpler elements, as well as the amount of memory available.

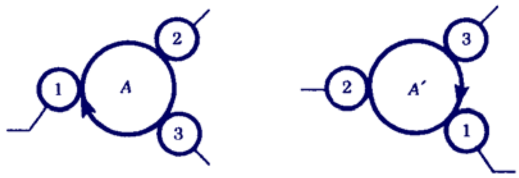

FIG. 66.

Some people may prefer to use a less geometric notation (e.g., preferring a matrix). There is nothing to prevent any notation being used, as long as its mapping from another, already existing system of notation can be summarized in a relatively few conversion rules. Indeed, a multiple display of a number of notations may turn out to be most useful, for which type of display is most useful depends, presumably, not only on the personal preferences of the user, but also on the particular activity in which he is engaged. Thus, a user may desire a rapid and convenient switch back and forth among notations or types of display and types of input media.

Automatic Sentence Samples

The generation of samples sentences is one such mapping that could be helpful. Starting with a relatively few sentences (say, S42, S43, S44) and their equivalents, as for example those summarized in Table 1 for S42, we should be able to convert fairly easily from one form to another. Clearly, the identities would introduce multiple values for each variable of a sentence and rules for the conversion of internal colligative terms (how many primes to use for complex triadas?) would have to be formulated.

Boundary and Internal Connection Conditions

To keep the structures generated well-formed, we can permit neither inconsistent relative ordering of arms passing through an 'envelope' nor can we tolerate the colligative connection of arms whose orders do not match. If we monitor these requirements automatically and mark inconsistencies (as by redrawing the offending envelope with a flashing or a dotted line, for example), the user of such an interactive system would benefit considerably by having his attention called at once to an ill-formed construction.

For internal connectives the same sort of alert would be helpful. Numbering of colligative terms depends on the relative order of the arms as joined; thus a colligative term cannot be unambiguously numbered if internal consistency is violated.

We can imagine more elaborate systems in which consistency, internal and boundary, is attempted automatically whenever any element of a graph is altered. Such a system would facilitate enormously the rapid and orderly development of a triadic calculus. Its application to sentence manipulation would be similarly much more direct.

Postscript

Riding another man's hobby-horse is sometimes a surprising experience. Warren McCulloch cajoled me into climbing on one of his. After falling off it numerous times (as we both expected), I found the triadic beast suddenly galloping off on its own (which I, at least, did not expect). This paper is a report of that brief but exhilarating ride.

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