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Logical Fiberings and Polycontextural Systems

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Abstract

Based on the notion of abstract fiber spaces the concept of a logical fibering is developed. This was motivated by a project where so-called polycontextural logics were discussed. The fiber space approach provides a rather general framework for the modeling of such non classical logics. It gives the possibility to construct a variety of new logical spaces from a given (indexed) system of logics. These spaces are in some sense parallel (inference) systems. We can give a straight forward definition and classification of the so-called transjunctions arising in polycontextural logics. These are bivariate operations having values distributed over different logical subsystems. Univariate, bivariate operations are introduced in functional notation. The group generated by the generalized negation operations and system changes is investigated. We make some remarks on aspects of applicability and links to other work.

1 Introduction

The following work was initiated by a joint project of two university groups and an industrial company on so-called 'Polycontextural Logic', abbreviated PCL. It is of importance for the whole understanding to give some motivating background information, we do this subsequently and in the next section when dealing with basics from PCL.

The PCL approach to a nonclassical generalization of two valued logics in form of a whole system of classical logical spaces distributed over an indexing set of values is heavily influenced by the philosophical work of Gotthard Günther (cf. references to PCL) and it cannot be seen without these roots. G.Günther's work has been extensively studied and partly continued by R.Kaehr and coworkers (we call them 'PCL group', for short).

One of the main arguments of the PCL group was that this so-called 'transclassical logic' should be suitable as a logical basis for modeling of (living) communicating systems. In fact, parts of that theory had been discussed and developed at the Biological Computing Laboratory (BCL), Urbana Ill., in the sixties, in the realm of research done to establish a new 2nd order cybernetics which requires a new logical basis (as was argued).

Many unconventional philosophical and metaphysical considerations can be found around the whole PCL theoretic approach. And it is not always easy to follow or adopt these thoughts. Unfortunately, much of the literature on the subject is not easy to access or available.

Therefore, one aspect of this article is to draw attention to some of these ideas and results and also to our own formal mathematical approach in the field of abstract fiberings which shows, among others, that PCL systems can be derived as a special class of logical fiberings. It should be emphasized here that I am a non-specialist in PCL and not a member of the PCL group.

Of further interest would be possible links to other work (e.g. in the field of the project MEDLAR (ESPRIT BRA 3125) on methods of practical reasoning), in particular to labelled deductive systems (cf. [*GA*]).

In many discussions during that project with the PCL group, intuitively, I always had the impression that a mathematical formulation of such distributed logical systems can be given naturally in a general framework using categories, fiberings, indexed systems (and related fields) as a formal mathematical language.

For example, in this way it is possible to give a simple definition of the notion of *transjunctions* (this is a typical nonclassical bivariate operation in PCL systems) and their classification. On the basis of the

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notion of an abstract fibering we establish a method to construct (many valued) logical spaces from a given (indexed) system of 2-valued logics.

This way, it is possible to derive a variety of new logical spaces systematically and it is easy to examine situations where formulas are consistent locally (in each subsystem) but not globally.

We apply the construction method to show that PCL systems can be derived as a special class of logical fiberings. A particular example in a 3-valued PCL motivates, more general, that the fibering approach leads to a method for decomposing (parallelizing) a given multiple-valued logic into 2-valued components (this is subject for further study).

The following presentation is not as rigorous as it could be since there are natural links to disciplines like indexed categories, toposes and sheaves which provide a more general framework for all such considerations (cf. selected titles in the literature list). A more general treatment can be subject of future work.

Further comments follow in the subsequent text. We also make a remark on intended possible applications in robot multitasking problems and possible links to other projects.

More details of the material presented in this article can be found in [$PF3$].

2 Remarks on PCL

We give some preparatory comments which, of course, can only represent a very limited perspective.

As previously remarked, PCL arose from particular philosophical considerations and has its individual understanding of communication and interaction. For a thorough understanding of the arguments of the PCL group it is necessery to have some insight in the written work.

A lot of material (case studies, reports) exists where ideas of polycontexturality are applied to nonformal, descriptive modeling of processes and scenarios (cf. e.g. [$R1, 2$]).

Basic principles are among others:

Distribution of several classical (2-valued) logics ('loci'). At least 3 loci are involved; the individual spaces are pairwise isomorphic ('locally'). For two classical spaces (components) of the whole system a third one has the function of *mediation*. Thus in certain respect, a pair of local components of the polycontextural system needs a third classical space for 'reflection' (for mediation) in the general communication process and in interaction. Although two components are isomorphic as 2-valued logical spaces, their placement ('index') plays a role in the common context of the whole system - the whole PCL system is multivalued. The 'transitions' ('communication') between the particular subsystems is of essential structural importance. Each individual subobject conceives the world through the same logic, but from a different place in the system; locally the results are all the same since the same 2-valued logic is placed there; but globally there may be differences in the results since reasoning is performed at different ontological places, the places being enumerated (labelled) by an index set; the role of the latter is twofold also constituing a global set of values for the PCL as a multivalued logic. This involvement is somewhat subtle (we refer to the literature on PCL).

The development of PCL as initiated by G. Günther is deeply influenced and based on philosophical considerations and it cannot be seen without these roots (we refer to the literature; cf. also some selected quotations from PCL literature in [$PF3$]).

We point out that we are neither specialists in these philosophical foundations nor experts in PCL as represented by the PCL group.

We find some aspects of this work quite interesting and were motivated to model such PCL-systems as certain logical fiberings.

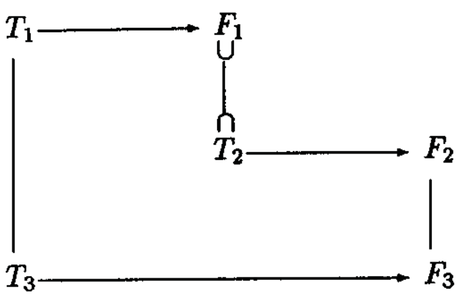
In this very general framework the PCL systems form a special class of multiple- valued logical spaces representable as certain fiberings.

3 Some Basic Notions from PCL

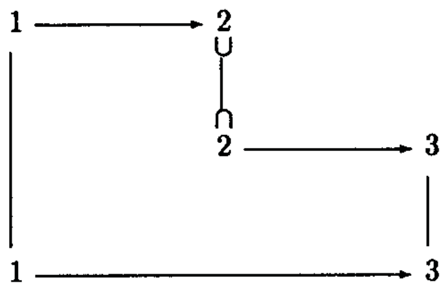
We use L or L_i (if an index is necessary) to denote a classical 2-valued logical space (a 1st order language w.r.t. a symbol set which we do not explicitly specify if there is no need to do so).

3.1 Local and Global Systems

A PCL system is an m-valued logical system consisting of n classical 2-valued subsystems denoted by L_i, $i = 1, 2, \ldots, n$, where $n = \binom{m}{2}$. The (global) truth values are denoted by $1, 2, \ldots, m$. (Thus each subsystem L_i can be associated with a 2-element subset of $\{1, 2, \ldots, m\}$.) The two (local) truth values within the (classical) subsystem L_i are defined by $\Omega_i = \{T_i, F_i\}$.
The total (global) system is denoted by $\mathcal{L}^{(m)}$, where $\mathcal{L}^{(m)}$ can be seen as the disjoint union (coproduct) of sets: $L_1 \amalg \ldots \amalg L_n$.
In addition to these basic constituents of a PCL the following so-called *mediation scheme* - we write *MS* for short - is an essential data for the definition of a PCL.
We show such a scheme for the case $m = 3$ (hence $n = 3$): Notation *MS3*



resp.



It contains the following information:
The arrow $T_i \rightarrow F_i$ expresses an ordering of the two values within the subsystem L_i, and
$\supset\!\!-\!\!-\!\!\subset$ expresses the fact that an F-value in one system (L_1) becomes a T-value in another (L_2) (a "change" of truth values when changing the corresponding subsystems)- i.e. a 'semantical change'.
The vertical lines have to be interpreted as identifications.
The right diagram is a short notation where the global values are inserted (indicating the relations(identifications) between the corresponding local values).

Thus, the *MS* describes the global relations between the local values and contains informations about what happens if one passes from one subsystem to another. It expresses how the collection of the value sets $\Omega_i = \{T_i, F_i\}$ of the individual L_i becomes the global set of values $\{1, 2, 3\}$, respectively.
In logical fibering notation (cf. section 4) we shall express this by an equivalence relation on the union Ω^3 of all the local value sets $\Omega_i = \{T_i, F_i\}$. From the set of all local values the global value set is then obtained as a set of residue classes: $T_1 \equiv T_3, \quad F_1 \equiv T_2, \quad F_2 \equiv F_3$.
If we denote the equivalence class of T_1 by $[T_1]$, etc., then the three 'global' values are $1 = [T_1] = [T_3]$, $2 = [F_1] = [T_2]$, $3 = [F_2] = [F_3]$, corresponding to the foregoing mediation scheme.

REMARK. In PCL a certain enumeration convention for the subsystems is defined. For a motivation of this kind of indexing and enumeration as well as the mediation scheme we refer to the particular literature on PCL. We do not go into these details here (cf. also [$PF3$]).

NOTATIONAL CONVENTION. In PCL 'vector-like' formulas are studied. Hence, for terms, formulas (expressions) in a PCL vector notation is used: $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, etc., x_i corresponds to expressions in subsystem L_i , respectively. We shall make no dictinction between column and row notation of expressions X,Y, ...

3.2 Negations in a PCL

Particular negation operations are introduced in PCL via tableaux. These univariate operations consist of negations in particular subsystems combined with system changes. For more details we refer to the PCL literature (cf. also [$PF3$, section 3]). With our approach we give a general investigation of negations and system changes and describe the group they generate (section 5).

3.3 Bivariate Operations, Transjunctions

For notational simplicity, again we restrict the considerations to $\mathcal{L}^{(3)}$.
The tableau method is also used to introduce bivariate operations. Since every subsystem is a classical first order system we are led to bivariate operations which are defined componentwise, as for example:
$X \wedge \vee \wedge Y$ is to be interpreted in $\mathcal{L}^{(3)}$ as the operation where a conjunction is performed in the subsystem L_1 a disjunction in L_2 and a conjunction in L_3. Analogously the operation $X \wedge \vee \rightarrow Y$ has to be understood.

In vector notation: $X \wedge \vee \wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \vee y_2 \\ x_3 \wedge y_3 \end{pmatrix}, X \wedge \vee \rightarrow Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \vee y_2 \\ x_3 \rightarrow y_3 \end{pmatrix}.$

Question: can all such operations be formed consistently ?
Answer: it turns out that this is not the case in general, e.g. $X \wedge \vee \wedge Y$ can be formed but $X \wedge \vee \rightarrow Y$ cannot be defined consistently in $\mathcal{L}^{(3)}$. This will become clear when we consider semantical aspects and evaluation of such expressions.

The identification of certain truth values plays an important role from a semantical point of view and it impacts the introduction of bivariate operations. We will come back to this later.

We give an example of a formula: $N_1(N_1X \wedge \wedge \wedge Y) = X \vee \wedge \wedge N_1Y$.
For a typical PCL like proof by tableaux we refer to the literature (cf. also $[PF3, 3.11]$).
In section 6 we shall give a short proof of that formula in an operational way using our notation.

REMARK. In a PCL system a new type of binary operation arises: an operation where the four output values w.r.t. the four inputs in a local subsystem L_i are distributed over other subsystems $L_j, j \neq i$. These (non classical) operations are also introduced via tableaux in the PCL literature, they are called *transjunctions*.
Dealing with logical fiberings, it is easy to see how to define and classify such operations, cf. section 6, 7.

3.4 Remark on Implementations

Some rules for forming PCL formulas of the above type have been implemented in *Prolog* during the previously mentioned project and were used to verify some formulas.

4 Logical Fiberings

All categories are assumed to be small categories (object and morphism classes are sets).
Subsequently, we introduce the concept of abstract fiber spaces in great generality. There is much material in the literature showing that fiberings provide a powerful tool - a 'formal mathematical language' where local-global relations of objects and data are expressible.
(For example, in $[PF2]$ we made practical experiences with that concept when we applied geometric fiberings to solve some open problems which formerly existed in a category of geometric spaces).

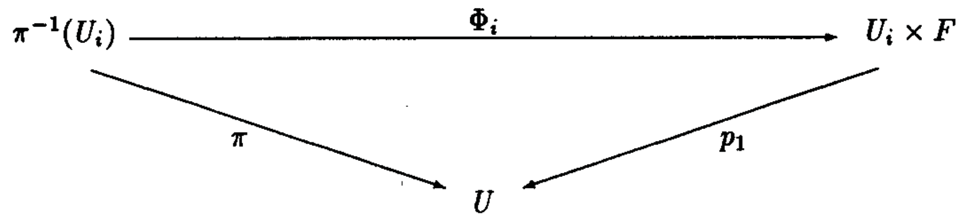
4.1 Preliminary Remarks on Indexed Systems

Fiberings and indexed systems are closely related from a formal point of view. As pointed out by P. Taylor in $[LNCS, p. 449ff]$, all consistency problems in forming families of sets w.r.t. a given indexing set I can be avoided when we interpret an indexed system $(A_i)_{i \in I}$ in terms of an abstract fibering ξ with a canonical projection (so-called ***display map***) π in the following sense:
the "total space" A of the fibering $\xi = (A, \pi, I)$ is the coproduct (i.e. disjoint union) of the A_i, hence $A = \coprod_{i \in I} A_i$, and $\pi(a) = i$ for all $a \in A_i$, defines the projection map $\pi : A \rightarrow I$ from A to the "base space" I. Then $A_i = \pi^{-1}(i)$ is exactly the fiber over i.

After these remarks we come to our general definition.

4.2 Fiber Spaces

We define a *fiber space* (*fibering*, *bundle*) $\xi = (E, \pi, B, F)$ in a very general way for objects of a category (not only for topological spaces).
The map $\pi : E \to B$ is sometimes called *projection*, E is called *total space*, B the *base space*, the set of all preimages of an element $b \in B$, i.e. $\pi^{-1}(b)$ is called *fiber over b*.
F denotes the *typical fiber* with which each fiber $\pi^{-1}(b)$, $b \in B$, of the bundle is modeled.
A *covering* $\{U_i\}_{i \in I}$ of the base space B consists in general of subsets of B (whose union is B); depending on the category additional properties and conditions can be required (e.g. that they are open sets).
Typically, a fibering ξ is locally trivial (w.r.t. $\{U_i\}_{i \in I}$), i.e. the following diagram is commutative:



Φ_i is an isomorphism in the corresponding category, where

$$\Phi_i = (\pi, \phi_i), \qquad \phi_i : \pi^{-1}(U_i) \to F \qquad \text{(a morphism).}$$

For $b \in U_i$, $\phi_{i,b} : \pi^{-1}(b) \xrightarrow{\cong} F$ is the fiber isomorphism induced by Φ_i (through this π^{-1} obtains its fiber structure).

The remaining properties concerning structure group, cocycle condition, etc. (cf. literature, e.g. [S]) can be formulated analogously here. We do not go into further details here since we only need the elementary features of the fiber space concept.

Of particular importance for our purposes is the case where the covering consists of one-point sets, i.e. the U_i are $1-$point sets (so every point of the base set B is an individual covering set).
The structure isomorphisms of the fibers are then given by:

$$\phi_b : \pi^{-1}(b) \xrightarrow{\cong} F$$

and for $b \neq c$ the fibers over b resp. c can be compared with each other (*fiber transition* with F as "mediator") using

$$\phi_c{}^{-1} \circ \phi_b : \pi^{-1}(b) \to F \to \pi^{-1}(c).$$

4.3 Logical Fiberings

A *logical fibering* is an abstract fiber space ξ as defined above where the typical fiber is $F = L$, in our considerations L will be a classical first order logical space.
The base space will often be denoted by $B = I$, the "indexing set".
We can also think of the fibers as modeled via a *boolean algebra* as typical fiber. Then ξ would be an *abstract bundle of boolean algebras*.
In this article we shall deal only with coverings where each U_i is a one point set, i.e. $U_i = \{i\}$ for $B = I$, the base space B of the fibering being the indexing set I.
A *morphism* between two logical fiberings is defined in a similar way as this is done for bundles (c.f. e.g. [$GO, Ch.$4.5], [LS] and others). In some sense we tend to interpret a morphism as a process of transporting information between spaces.
In this way we obtain the *category of logical fiberings*. Logical fiberings over a certain constant base space I (index set) then form a 'comma category' $\mathcal{L} \downarrow I$ in the usual sense (cf. e.g. [GO]).

REMARK. Although we are working here with classical logics as fibers, we point out that all considerations can be done for more general objects in the fibers (e.g. different logics or algebras, etc.), based on a

modified, generalized definition of an abstract fibering.
In a logical fibering the map $\pi : E \rightarrow I$ is always a morphism in the category of sets. The base set I can carry an additional own structure (e.g. partial order, graph, net, semigroup, algebra, topology, etc.).
If E, I belong to the same category, then it is reasonable to require that π is a morphism in that category. In particular, it might be of interest in this framework to study logical fiberings which are bundles or sheaves of e.g. boolean algebras over a topological base space.
We want to mention here that the fibering approach reflects certain internal parallelism.

NOTE. Although we introduced the notions (fiberings, etc.) in great generality our interest in this article concentrates only on a certain class of logical fiberings w.r.t. one-point coverings. We also do not discuss specific structures of the base space (index set). But nevertheless, we wanted to introduce the notions in a certain generality for later use. (These aspects are of interest for further study).

4.4 Free Parallel Systems

The simplest form of a fibering or bundle is the "trivial fibering" $\xi = (E, \pi, B, F)$ with $E = B \times F$, π the first projection; the fiber over $i \in B$ is: $\pi^{-1}(i) = \{i\} \times F$.

In the context of our logical fiberings such a trivial fibering is a *parallel system of logics* L_i over an index set I as base space B and $F = L$ a classical first order logic.
We can think of reasoning processes running in parallel and independently within each fiber $L_i = \pi^{-1}(i)$.
Transition ("communication") between fibers (loci) is described via the $\phi_i, i \in I$.
We call such a logical fibering a *"free parallel system"* $\mathcal{L}^I$. Its total space is denoted by E^I, note $E^I = \coprod_{i \in I} L_i$.
Simplest case (trivial fibering): all $\phi_i = id_L$. We shall make a difference between *local truth values* $\Omega_i = \{T_i, F_i\}$ in each 2-valued subsystem $L_i, i \in I$, and the set of *global values* Ω^I of the whole fibering.
Parallel systems are characterized by the fact that there are no relations between different local values, i.e. the global value set is the mere coproduct (disjoint union)

$$\Omega^I = \coprod_{i \in I} \Omega_i$$

("free" parallel system). For $I = \{1, \ldots, n\}$, n a natural number, we define $\mathcal{L}^n := \mathcal{L}^I$. The corresponding total space is denoted by E^n.

NOTATION. A logical fibering which is derived from a free parallel fibering $\mathcal{L}^I$ by the introduction of an equivalence relation $\equiv$ on Ω^I will be denoted by $\mathcal{L}^{(I)}$, with $\Omega^{(I)} := \Omega^I / \equiv$. For $I = \{1, \ldots, n\}$ we use the notation $\mathcal{L}^{(n)}$. Accordingly, the total space of such a fibering is $E^{(I)}$ or $E^{(n)}$, respectively.
All the logical operations we are considering in a logical space $\mathcal{L}^{(I)}$ are induced by corresponding operations in $\mathcal{L}^I$. In a certain sense, $\mathcal{L}^{(I)}$ is a logical fibering with constraints (cf. section 7 for more details).
For example, the PCL system $\mathcal{L}^{(3)}$ arises from the free parallel system $\mathcal{L}^3$ by introducing the equivalence relation $\equiv$ (given by *MS3*, cf. section 3) on the global values Ω^3 yielding $\Omega^{(3)} = \{1, 2, 3\}$.

REMARK. In the free parallel logical systems $\mathcal{L}^I$ there are no restrictions on the value sets Ω^I (e.g., in $\mathcal{L}^n$ there is a total amount of $2n$ global truth values). A variety of logical systems can be derived from $\mathcal{L}^I$ by introducing various equivalence relations on Ω^I. We can vary freely all data, i.e. base space, the fibers. In principle, we can combine different logics in a fibering when we allow different types of logical spaces for the fibers (by a corresponding generalization of the definition of an abstract fibering.)
If we consider (in the category of sets) the total space E^I of the fibering $\mathcal{L}^I$ as coproduct of the sets $L_i (i \in I)$, then $\mathcal{L}^I$ is a bundle over I, i.e. an object of the comma category Set$\downarrow I$, also denoted Bn(I), bundles over I, cf. $[GO, Ch.4.5]$.
This category actually is a *topos*, $[Goldblatt, loc.cit.]$. We do not go into these details here.

4.5 Notation for Logical Expressions

With the notion of a logical fibering we express the coexistence of various logical loci residing over an indexed system (base space) which itself can have an own structure (object of certain category). Our objective is

to give a suitable formalization of the logical expressions in a fibering $\mathcal{L}^I$ which form the corresponding language of $\mathcal{L}^I$. This should be constructed from the local languages of the $L_i, i \in I$. In a free parallel fibering it is possible without restrictions to form global expressions consisting of a family of arbitrary local expressions (formed in parallel in each subsystem).
Canonically, this leads to the following formal *definition of a global expression* x in the language of $\mathcal{L}^I$, namely $x = (x_i)_{i \in I}$.
The language of a logical fibering is therefore obtained as the collection of all families of expressions from the local languages of the subsystems L_i.
Formally, all such families $x = (x_i)_{i \in I}$ form the *direct product* (in the categorical sense) of the sets of all local expressions, we denote this by $\prod_{i \in I} L_i$.
Logical connectives are introduced componentwise.
If I is a finite set then we use vector notation for expressions x, as already done previously.
We mention here that, alternatively, the set of all global expressions can be expressed as the set of all *sections* $s : I \to \coprod_{i \in I} L_i$, a section has the property $\pi \circ s = id_I$, π being the projection of the fibering. This is more compatible with our notion of a fibering (cf. P.Taylor in [$LNCS$, p.451] and also [$PF3$]).
NOTATION. We use the symbol E^I also to denote the language corresponding to the logical fibering (keeping in mind that it is a direct product or all sections, respectively).

5 Univariate Operations, Negations

5.1 Preliminary Remark

The negation operation N_1 in the PCL $\mathcal{L}^{(3)}$ is defined (cf. e.g. [$PF3$]) in such a way that N_1 realizes a classical negation in L_1 and swaps the contents of the places L_2 and L_3 (system change) — i.e. realizes the transposition $2 \mapsto 3$, $3 \mapsto 2$ (with cycle description for permutations, briefly written as (23)). This can be represented as follows:
$N_1 : \mathcal{L}^{(3)} \to \mathcal{L}^{(3)}$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{N_1}{\longmapsto} \begin{pmatrix} \overline{x_1} \\ x_3 \\ x_2 \end{pmatrix}, i.e. \qquad N_1 X = N_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \overline{x_1} \\ x_3 \\ x_2 \end{pmatrix}$$

Such operations can be canonically extended to general logical fiberings.

5.2 Transpositions (System Changes)

Let $\mathcal{L} = \mathcal{L}^n$ (the following considerations can easily be generalized to $\mathcal{L}^I$).
Permutations play an important role in the definition of negation operations in PCL.
We recall that every *permutation group* (i.e. the group of all bijections of a set onto itself — here we consider mainly finite sets) can be generated by *transpositions*.
A transposition swaps two elements (numbers) and leaves the rest fixed, e.g.

$$\begin{pmatrix} 1 \ldots i \ldots j \ldots n \\ 1 \ldots j \ldots i \ldots n \end{pmatrix} = (ij)$$

(ij) is the cycle notation for permutations. We let permutations operate on the indices of the subsystems $L_1, \ldots, L_n$ and can then describe *system changes* as follows:
we denote by $\tau_{ij} : \mathcal{L} \to \mathcal{L}$ the transposition (system change)

$$\tau_{ij}(X) = (x_1, \ldots, x_{i-1}, \phi_{ij}(x_j), \ldots, \phi_{ji}(x_i), x_{j+1}, \ldots, x_n),$$

which means: the expression x_j in the place (fiber) over j is transferred to L_i by ϕ_{ij} (fiber transition isomorphism) and,conversely, the content of position i is brought to the fiber (logical place) L_j by means of ϕ_{ji}.
This corresponds to a "system change" by means of $\phi_{ij}, \phi_{ij}^{-1}$.
Note that for evaluations $w_i : L_i \to \Omega_i$ and $w_j : L_j \to \Omega_j$ it holds that: $\quad w_i(\phi_{ij}(x_j)) = w_j(x_j)$.
To shorten notation we shall omit the ϕ_{ij} in our formulas and abbreviate $\tau_{ij} = (ij)$, if no confusion arises.

5.3 Negations in Subsystems

By a "*local*" or "*inner*" *negation* we mean one of the following operators:
$n_i : \mathcal{L} \to \mathcal{L}$, $n_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, \overline{x_i}, \dots, x_n)$, for $i = 1, \dots, n$.
So, only in the system L_i the negation is applied, all other places remain unchanged.
It holds that $n_i \circ n_j = n_j \circ n_i$, *for* $i \neq j$ and $n_i \circ n_i = Id_{\mathcal{L}} =$ the identity on $\mathcal{L}$.
Now we can compose negation operations on $\mathcal{L}$:
The *negation operator* $N_{ij}^k : \mathcal{L} \to \mathcal{L}$ is defined as the composition of the operators n_k and τ_{ij}, namely $N_{ij}^k = \tau_{ij} \circ n_k$.
So, first a negation is carried out in the subsystem L_k and then the expressions on the positions i and j are interchanged (by τ_{ij}).

EXAMPLE. Negation operations in $\mathcal{L}^3$:
With the above notation we obtain the following operators ("global" negations) on the fibering $\mathcal{L} := \mathcal{L}^3$ by composition of elementary operations.
(The same can be done by passing to $\mathcal{L}^{(3)}$, but well-definedness problems have to be handled with care).
In particular we obtain the negation operations $N_1, \dots, N_5$ using the notational convention of PCL (cf. $[R1], [R2], [PF3]$):

$$N_1 = n_1 \circ (23), \quad N_2 = n_2 \circ (13), \quad N_3 := N_2 \circ N_1, \quad N_4 := N_1 \circ N_2, \quad N_5 = N_1 \circ N_2 \circ N_1$$

Furthermore, $n := n_1 \circ n_2 \circ n_3 : \mathcal{L} \to \mathcal{L}$ (negation in each corresponding subsystem).
For every transposition $\tau \in \{(12), (13), (23)\}$ it holds: $\tau \circ n_i \circ \tau^{-1} = n_{\tau(i)}$ *where* $\tau = \tau^{-1}$, (i.e. τ operates on the elementary (inner) negations n_1, n_2, n_3 by conjugation).

Summarizing the previous considerations, we see that the negation operations and system changes generate a group. We do this here only for the particular example $\mathcal{L}^3$ and remark that this result also holds in the general case.

Let $\mathrm{N} =< n_1, n_2, n_3 >$ be the group of operators on $\mathcal{L}$ generated by the ("inner", "local") negations n_1, n_2, n_3.
N contains 8 elements and is isomorphic to the "elementary abelian 2-group"

$$\mathrm{N} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

The transpositions τ_{ij} generate the group of "system changes", S_3, which is isomorphic to the full group of all permutations of 3 elements.
Combining the two groups we can define the group of *"global" negation operators* $\mathcal{N}$, which is generated by N and S_3: $\mathcal{N} :=< \mathrm{N}, S_3 >$.
Using the previous results it can be shown that $\mathcal{N}$ can be represented as a *semidirect product*:

$$\mathcal{N} := \mathrm{N} \rtimes S_3$$

(since it is possible to represent $\mathcal{N}$ as a product of groups $\mathcal{N} = \mathrm{N}.S_3$, $\mathrm{N} \triangleleft \mathcal{N}$ is a normal subgroup of $\mathcal{N}$ and S_3 operates on N by conjugation).
These considerations can be directly generalized and describe all the univariate operations we want to have in a (free parallel) logical fibering.

6 Bivariate Operations

6.1 Domain of Definition

Given a logical fibering $\mathcal{L}^I$ the question arises how to define bivariate logical operations, more precisely how does a suitable domain look like on which we can define an operation in a natural way, fitting to our fiber space concept.
In a natural way this leads us to the family of products $(L_i \times L_i)_{i \in I}$ where we can make a componentwise definition of a bivariate operation.
In a formally correct way and compatible with our fiber space notation such a family can be expressed by

the *pullback* (or fibered product) denoted by $E^I \times_I E^I$, cf. Taylor, in [*LNCS*, *p.449ff*]. See also [*PF*3, 6.1] for more details.
A bivariate operation on $\mathcal{L}^I$ should map each pair of expressions $(x_i)_{i\in I}, (y_i)_{i\in I}$ in $\mathcal{L}^I$ to a new expression of $\mathcal{L}^I$ (an image of this mapping). Equivalently, such a pair corresponds to the familiy of pairs $((x_i, y_i))_{i\in I}$ which is exactly an element of the pullback $E^I \times_I E^I$.

REMARK. In a free parallel system $\mathcal{L}^I$ many bivariate operations can be introduced componentwise combining various bivariate operations, defined independently on each component (subsystem) L_i, $i \in I$.

6.2 Examples

For the sake of brevity we consider examples in $\mathcal{L}^3$ combining negations and bivariate operations. We want to demonstrate the operational way in which the formalism works. As mentioned in the beginning, all operations are expressed by mappings (functional notation).

(a) We show $N_1(N_1X \wedge\wedge\wedge Y) = X \vee\wedge\wedge N_1Y$.
Proof:

$$N_1(N_1X \wedge\wedge\wedge Y) = N_1\left(\begin{pmatrix}\overline{x_1}\\ x_3\\ x_2\end{pmatrix} \wedge\wedge\wedge \begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\right) =$$

$$= N_1\begin{pmatrix}\overline{x_1}\wedge y_1\\ x_3\wedge y_2\\ x_2\wedge y_3\end{pmatrix} = \begin{pmatrix}\overline{\overline{x_1}\wedge y_1}\\ x_2\wedge y_3\\ x_3\wedge y_2\end{pmatrix} = \begin{pmatrix}x_1\vee \overline{y_1}\\ x_2\wedge y_3\\ x_3\wedge y_2\end{pmatrix} =$$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \vee\wedge\wedge \begin{pmatrix}\overline{y_1}\\ y_3\\ y_2\end{pmatrix} = X \vee\wedge\wedge N_1Y$$

The mixed expressions like $x_2 \wedge y_3$, etc. should be constructed via the transition isomorphisms ϕ_{23}, etc., but we omit this for short.
(b) Analogously (cf. [*PF*3, section 6]): $N_5((N_5X) \vee\vee\vee N_5Y) = X \wedge\wedge\wedge Y$.

6.3 Transjunctions

In a parallel system $\mathcal{L}^I$ the following situation arises naturally for bivariate operations: a local pair (x_i, y_i) in $L_i \times L_i$, $i \in I$, can be mapped into different subsystems $L_j, L_k, \ldots$.
With respect to the four possible local input pairs from $\Omega_i \times \Omega_i$ there are maximally four different subsystems for the images.
That means that such bivariate operations distribute images over different subsystems — *a new feature.*
Such bivariate operations are called *transjunctions.*
More details will be discussed in section 8 where we give a classification of transjunctions. It is helpful using evaluations to make things more transparent.

7 Evaluations, Semantical Aspects

7.1 Remark

As already remarked, starting from a free parallel system $\mathcal{L}^I$, there are many possibilities to find new logical spaces by introducing an equivalence relation $\equiv$ on the global value set Ω^I and then examining the passage $\mathcal{L}^I \to \mathcal{L}^{(I)}$.
Many multiple-valued logical spaces can be constructed systematically by this method.

Considering PCL systems, $\mathcal{L}^{(n)}$ is derived from $\mathcal{L}^n$ by the special $\equiv$-relation defined through the mediation scheme (cf. section 3).
From this point of view we obtain PCL systems as a particular class of certain logical fiberings.

To formalize the evaluation process we use an ad hoc notation which is useful for our purposes.

Again we point out that a rigorous formal treatment in categorial notions would be possible e.g. in the sense of Goldblatt $[GO, Ch.\ 6]$, cf. also $[LS]$.

7.2 Formalization of the Evaluation Process

In accordance with our notions of logical expressions and bivariate operations the (global) evaluation procedure w.r.t. a logical operation will be introduced componentwise hence being a family of local (classical) valuations.
We consider here only bivariate operations; evaluation of univariate operations can be introduced analogously.

Let Θ be a bivariate operation on $\mathcal{L}^I$. For every $i \in I$ let $w_i : L_i \to \Omega_i$ be a (classical) valuation and let $w := (w_i)_{i \in I}$ be the family of these valuations. This induces a global valuation $w : \mathcal{L}^I \to \Omega^I$.
Note, although we deal with families of expressions and truth values we do not express the domain and codomain of w as direct products explicitely (cf. similar remark at the end of section 4).

A global valuation $V(\Theta)$ of the operation Θ is defined componentwise by the following composition of maps (using functional notation).

$$V_i(\Theta) : \Omega_i \times \Omega_i \xrightarrow{J_i} L_i \times L_i \xrightarrow{\Theta} \mathcal{L}^I \xrightarrow{w} \Omega^I.$$

Where J_i denotes the local *input map* (substitution of pairs of truth values)

$$J_i : \Omega_i \times \Omega_i \to L_i \times L_i$$

which substitutes (x_i, y_i) by pairs of logical values in the expression $\Theta(x_i, y_i)$. The global valuation is then defined by the (family of the) $V_i(\Theta), i \in I$.

The evaluation procedure for a derived logical fibering $\mathcal{L}^{(I)}$ is induced by the previously described evaluation procedure in $\mathcal{L}^I$.
Let $p : \Omega^I \to \Omega^{(I)}$ be the canonical residue class map, where $\Omega^{(I)} = \Omega^I / \equiv$, for an equivalence relation $\equiv$ on Ω^I.
In this case, again, the evaluation is defined via the components, but we have to take into account the given $\equiv$ –relation and well-definedness properties (cf. the following example).
In formal notation, for $i \in I$, the *induced* valuation $V_{(i)}(\Theta)$ is defined by

$$V_{(i)}(\Theta) : \Omega_i \times \Omega_i \xrightarrow{V_i(\Theta)} \Omega^I \xrightarrow{p} \Omega^{(I)}$$

hence the local input is on pairs $\{T_i, F_i\}$ but respecting that these are representatives of equivalence classes and whenever two pairs (from different subsystems) belong to the same equivalence class the resulting value of $V_{(i)}(\Theta)$ has to be the same (this corresponds to well-definedness).
Thus we are led to certain constraints on the evaluation procedure.

REMARK. It is important to note again that the images of the four possible local input pairs of Θ can be distributed over maximally four different value sets $\Omega_\alpha, \Omega_\beta, \Omega_\gamma, \Omega_\delta$.

EXAMPLE. If we use 2×2 - matrix notation for the input pairs and corresponding output, the following is an example of a transjunction if $\{\alpha, \beta, \gamma, \delta\}$ contains at least 2 different indices.

$$\begin{array}{|cc|} \hline T_iT_i & T_iF_i \\ F_iT_i & F_iF_i \\ \hline \end{array} \quad \overset{V_i}{\longmapsto} \quad \begin{array}{|cc|} \hline T_\alpha & F_\beta \\ F_\gamma & F_\delta \\ \hline \end{array}$$

NOTATION. Subsequently, in all our considerations we express the value scheme (truth table) of a local bivariate operation by a 2×2 - matrix where only the output values (images of the operation) are represented (as in the right matrix above), their position (index pair) in the matrix is determined by the position of the corresponding input pair (cf. left matrix).

We use this convention analogously for 3-valued (multiple-valued) operations.

REMARK. We recall that we have chosen $(L_i \times L_i)_{i \in I}$ (the pullback) as domain of definition for a bivariate logical operation Θ on a free parallel system $\mathcal{L}^I$. This means that we do not consider input pairs $(x_i, y_j), i \neq j$, i.e. where x_i, y_j are from different subsystems (since we are mainly interested in forming logical connectives componentwise).
For a particular $\mathcal{L}^n$ the value set Ω^n totally contains $2n$ values.
A bivariate operation in the usual sense of multiple-valued logics would then be represented by a $(2n)\times(2n)$–value matrix. In our definition of Θ on $\mathcal{L}^n$ the operation is represented by n 2×2– submatrices which are arranged one after the other along the main diagonal of the whole $(2n) \times (2n)$– scheme (in this sense it represents a restricted map). For further remarks on this we refer to $[PF3]$.

7.3 Two Examples

We briefly discuss the following evaluation problem. Let Z be the bivariate operation $Z =$X $\wedge \vee \rightarrow Y$.
No consistency problem arises, of course, when we form Z in the system $\mathcal{L}^3$.
We recall that the global value set $\Omega^{(3)}$ of $\mathcal{L}^{(3)}$ is given by $1 = [T_1] = [T_3]$, $2 = [F_1] = [T_2]$, $3 = [F_2] = [F_3]$ (corresponding to the mediation scheme *MS3*).
The corresponding local evaluations can be expressed by the following 2×2- matrices (on the right side we have inserted the corresponding global values):

$$x_1 \wedge y_1 : \quad \begin{matrix} T_1 & F_1 \\ F_1 & F_1 \end{matrix} \equiv \begin{matrix} 1 & 2 \\ 2 & 2 \end{matrix}$$

$$x_2 \vee y_2 : \quad \begin{matrix} T_2 & T_2 \\ T_2 & F_2 \end{matrix} \equiv \begin{matrix} 2 & 2 \\ 2 & 3 \end{matrix}$$

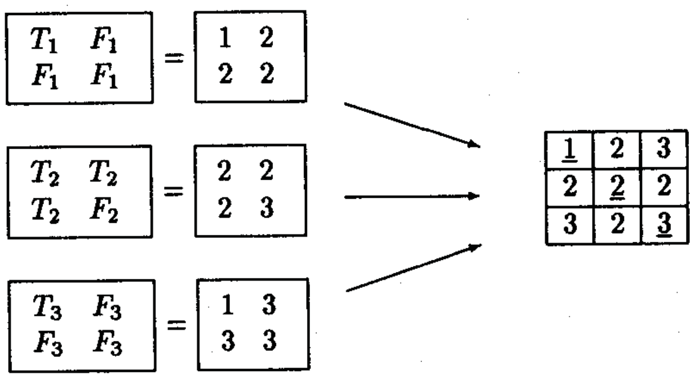
$$x_3 \rightarrow y_3 : \quad \begin{matrix} T_3 & F_3 \\ T_3 & T_3 \end{matrix} \equiv \begin{matrix} 1 & 3 \\ 1 & 1 \end{matrix}$$

Applying the valuation $V_{(i)}(Z)$ - as defined above - we have to respect the relations between input pairs $(T_1, T_1) \equiv (T_3, T_3)$, $(F_1, F_1) \equiv (T_2, T_2)$, $(F_2, F_2) \equiv (F_3, F_3)$ according to the above mentioned identifications of local truth values.
In the evaluation procedure these identities have to be respected, i.e. equivalent pairs lead to equal images. (Alternatively, this can be expressed by the condition that the above three 2×2-value matrices are composable into one 3×3-scheme ('morphogram'), cf. next example).
Evaluating $X \wedge \vee \rightarrow Y$ leads to inconsistencies, since

$$\begin{matrix} (F_2, F_2) \stackrel{\vee}{\longmapsto} [F_2] = 3 \\ \equiv \qquad\qquad \neq \\ (F_3, F_3) \stackrel{\rightarrow}{\longmapsto} [T_3] = 1 \end{matrix}$$

(Locally consistent, but not globally).
The pair (F_2, F_2) is local input in the second subsystem and (F_3, F_3) in the third, respectively. But globally, as input of the valuation of the bivariate operation Z on $\mathcal{L}^{(3)}$ both pairs are equal (since they belong to the same equivalence class). That means they have to produce the same image (output) of that operation (in the sense of a mapping).

APPLICATIONAL ASPECTS. It might be an interesting aspect whether such situations can be exploited to model specific applications where certain local operations are prohibited - from a global perspective (evaluation).
In the above case the implication in the third subsystem causes problems.
When we consider for example $X \wedge \vee \wedge Y$ these problems do not occur:



REMARK. The diagram on the right is an amalgamation of the three $2 \times 2-$ matrices on the left, suggesting that consistency is expressible in forming such a condensed form. The three $2 \times 2-$ submatrices along the diagonal have to be compatible in such a way that coinciding diagonal elements have to be equal (compatibility with the $\equiv$ – relation).
All the three 2×2-value schemes are represented (merged) in the 3×3- value scheme as submatrices. In this form the operation $X \wedge \vee \wedge Y$ in $\mathcal{L}^{(3)}$ is represented by the complete 3×3- matrix like a 3-valued logical connective, cf. [$PF3$].

Reversing this procedure leads to a method for decomposing a bivariate operation (given by a corresponding value matrix) in a multiple-valued logic into a system of 2-valued operations. In a certain sense this can be interpreted as a *parallelization* method for multiple-valued logics.
This will be subject of another work (forthcoming preprint in RISC-Linz publication series).

8 Classifying Transjunctions

For the classification of transjunctions it is convenient to consider the relevant evaluation procedures.

Let $\mathcal{L} := \mathcal{L}^I$, $\Omega := \Omega^I$ and Θ be a bivariate operation (actually we are interested in transjunctions).
For a local subsystem $L_i, i \in I$, we consider $\Theta : L_i \times L_i \to \mathcal{L}$ and w.r.t. $V_i : \Omega_i \times \Omega_i \to \Omega$ we can represent Θ locally by a 2×2 *pattern* (called morphogram in PCL notation), cf. the examples in section 7.
Suppressing the 4 indices $\alpha, \beta, \gamma, \delta$ in that *T,F-pattern* we obtain one of the sixteen $2 \times 2-$ value patterns corresponding to bivariate operations of classical (1st order) logic.

Using this, a transjunction can be described by such a $2 \times 2 - T, F -$ ***pattern*** followed by a ***distribution*** of the $T, F -$ ***values*** over (maximally four different) value sets $\Omega_\alpha, \Omega_\beta, \Omega_\gamma, \Omega_\delta$ corresponding to subsystems $L_\alpha, L_\beta, L_\gamma, L_\delta$.

More formally, let ϑ denote a classical bivariate operation $\vartheta : L_i \times L_i \to L_i$ and let $V = (w_i)_{i \in I}, w_i : L_i \to \Omega_i$ be valuations.

For $(T_i, F_i) \in \Omega_i \times \Omega_i$ let $\chi_{(T_i,F_i)} : \Omega_i \times \Omega_i \to \{0,1\}$ be the corresponding characteristic function.
Then the local evaluation of the transjunction $\Theta : L_i \times L_i \to \mathcal{L}$ can be described by

$$
\begin{aligned}
\mathrm{w}_{\alpha\beta\gamma\delta}(,) = \quad & \chi_{(T_i,T_i)}(,) \cdot w_\alpha \phi_{\alpha i} \vartheta(,) \quad + \\
& \chi_{(T_i,F_i)}(,) \cdot w_\beta \phi_{\beta i} \vartheta(,) \quad + \\
& \chi_{(F_i,T_i)}(,) \cdot w_\gamma \phi_{\gamma i} \vartheta(,) \quad + \\
& \chi_{(F_i,F_i)}(,) \cdot w_\delta \phi_{\delta i} \vartheta(,) \quad .
\end{aligned}
$$

We recall that the ϕ_{jk} denote the system changes (cf. section 4).

This local evaluation $w_{\alpha\beta\gamma\delta}$ could also be expressed by a map $D_{\alpha\beta\gamma\delta} \circ V_i$, with $V_i = w_i \circ \vartheta \circ J_i : \Omega_i \times \Omega_i \to \Omega_i$ and $D_{\alpha\beta\gamma\delta} : \Omega_i \to \Omega$ ***distributes values*** over different subsystems in the following way:
let for example F_i be the first value in the $2 \times 2-$ value matrix belonging to ϑ, then $D_{\alpha\beta\gamma\delta}(F_i) = F_\alpha$, analogously, if the third value would be T_i, then $D_{\alpha\beta\gamma\delta}(T_i) = T_\gamma$, etc..
In other words $D_{\alpha\beta\gamma\delta}$ transforms the $2 \times 2-$ value matrix of ϑ by substituting the indices $\alpha, \beta, \gamma, \delta$ (in this order) for the corresponding $T, F-$ values (cf. e.g. the following figure).

$$\begin{array}{|cc|}\hline T_i & F_i \\ F_i & F_i \\ \hline\end{array} \rightarrow \begin{array}{|cc|}\hline T_\alpha & F_\beta \\ F_\gamma & F_\delta \\ \hline\end{array}$$

This yields a systematic way to classify transjunctions in $\mathcal{L}^I$.

NOTATION. We can speak of conjunctional, disjunctional, implicational, ..., transjunctions corresponding to whether ϑ is a conjunction, disjunction, implication, etc., since the $T, F-$ value matrix of ϑ characterizes the transjunction type.

REMARK. Transjunctions extend the set of bivariate operations extensively.
Passing to $\mathcal{L}^{(I)}$ we have to respect well-definedness problems, similar to the example discussed in section 7. Therefore, the possibility for forming bivariate operations which involve transjunctions depends on the structure of the set of global values $\Omega^{(I)} = \Omega^I / \equiv$.
We are interested in possibilities to apply transjunctions in practical fields like robotic scenarios, for example.

9 Concluding Remarks

Concerning *labelled deductive systems (LDS)*, D.Gabbay pointed out that various structures for the label systems are of interest, e.g. semi groups, boolean algebras. Besides that it is a very interesting question whether there are possible links to LDS or possibilities to combine certain features.

We are in particular interested in logical fiberings which are deduced from free parallel systems by a *group action* on the value set.

Remarks on the *base space structure*:
Different structures for the base space (indexing set) may be of interest, e.g. totally ordered sets (in the case of certain PCLs); partially ordered sets (objects of the category POSET); semi groups; net structures; and others.
Of particular interest can be an *ultra metric base space*, these spaces appear naturally in the study of hierarchical structures (cf. $[E]$: 1-1 correspondence between indexed hierarchies and ultrametrics).

REMARK. We pointed out that we do not use here the categorical language systematically as, e.g., in dealing with $Bn(I)$ and $Sh(I)$, the category of bundles and sheaves, respectively (cf. $[GO], [LS], [LNM1,2], [RB]$), although this possibility exists for our approach.
We prefer a less abstract formulation here for a first attempt to present the main notions.
We adopt a more *engineering point of view* in the sense that we suppose that such logical fiberings might be suitable tools in situations where indexed systems play a role and this arises frequently.
Therefore we are motivated by *practical reasons* rather than by philosophical or purely theoretical ones.

In general, we can say that the fibering approach allows many constructions since it is a very general 'formal mathematical language', in particular we think of topological and differentiable manifolds as base spaces - this might also be of interest from a physical point of view.

The most general categorial framework seems to be "Indexed Categories" (cf. $[LNM1]$). We refer also to the corresponding remarks in $[PF1]$, especially on the impact of the base space structure on the whole system.

REMARK. The generality of the fibering concept allows, in principle, to put different logics in the fibers over a common base space, that means to mix different logics. One has to describe carefully the transition ('communication') between different fibers.

APPLICATIONAL ASPECTS. Possible applications of the concept of logical fiberings can be seen from different perspectives. We find it interesting to try to apply it in the field of robotics, especially robot multitasking scenarios as they are discussed e.g. in the MEDLAR project.
For example, a space where actions are performed which are to be modeled formally (e.g. robots cooperating in a robot cell) may be covered by regions where each region has its own logic ('typical fiber'). This refers

to local triviality of an abstract bundle. Passing from one region to another causes a change of the logics applied; this is a certain local global interaction principle (typically included in the concept of fiber bundles and sheaves).
In particular, it is very interesting to find out possibilities how to apply transjunctions (some first ideas are in discussion).

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