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Further Towards a Triadic Calculus

Part 2

Christopher R. Longyear [*]

*Triadic Operations (or Relative Operations)**

Now we introduce the existential quantifier Σ: (read "there is some ...") and the universal quantifier Π (read "all," or "everybody" or "everything"). Application of a quantifier to a triada gives a lower structure (a structure with a lower number of blanks). For example, $H_{jk} = {}_iG_{ijk}$ reads

"there is some who gives __to__" (S29)

that is, a diadic structure. In order to obtain a closed operation, we could define an "open" or "external" product or sum to obtain a higher structure, and then reduce it to a triada, by applying one of the two quantifiers one or more times. For example, let "and" be the open operation between L_{ijk} and G_{emn} such that $L_{ijk} \bullet G_{emn}$ means

"__lies in between__and__, and__gives__to__" (S30)

that is, a hexada. If we now "contract" by application of the quantifier, we obtain the triada

$$M_{jkn} = \sum_{iem} L_{ijk} \bullet G_{emn} \qquad [E1]$$

This reads

"there is some individual who lies in between__and__, and someone gives something to__" (S31)

*end of **

The structures representing graphically the sentences (S29)-(S31) of this paragraph are shown in Fig. 16.

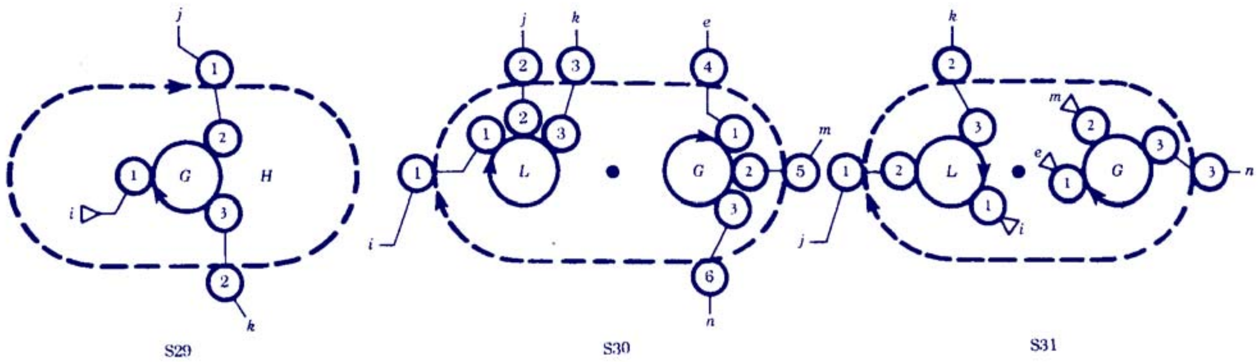

FIG. 16.

We note that these structures are not "well-formed" according to criterion 6 of p. 3 (part 1). Such complex structures therefore cannot be replaced by their envelopes, for the ordered sequence of the "arms" inside and outside an envelope do not match. Ill-formed interfaces are indicated by drawing them with broken rather than solid lines. If we recast Fig. 16 as Fig. 17 such that

$$H'_{ij} = \sum_{k} G'_{ijh}$$

reading

"__gives__to someone." (S32)

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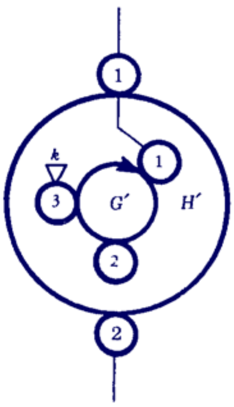


FIG. 17.

we have a well-formed dyad at the interface between the arms of G' and those of H'. Further examination may reveal that only the sequence, not the order as well need to be maintained, and that the structure of sentence (S29) is well-formed, while those of sentences (S30) and (S31) are not. The dyad H' = G' reads

"__gives__" (S33)

which is equivalent to (S32) if the sense of "give" is the same.

* More interesting are the combinations of triadas with some elements, or blanks, in common, that is, having colligative terms. Such is the case of the so-called relative products and sums for binary, or diadic, relations. For triadas, let us write the product with one colligative term

$$M_1 = M_{ijkem} = L_{ijk} \bullet G_{kem} \qquad [E17]$$

that reads

"__lies in between__and__who gives__to__" (S34)

that is, a pentadic structure. If we now contract upon the repeated index, by means of the quantifier, we obtain

$$M_2 = M_{ijem} = \sum_k L_{ijk} \bullet G_{kem} \qquad [E18]$$

that is, the tetrada

"__lies in between__and someone who gives__to__" (S35)

end of *

These sentences (S34) and (S35) are represented graphically in Figs. 18 and 19.

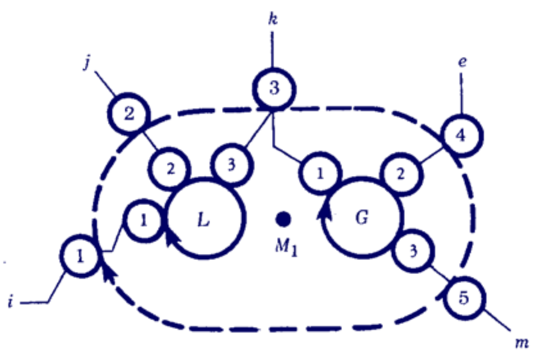


FIG. 18.

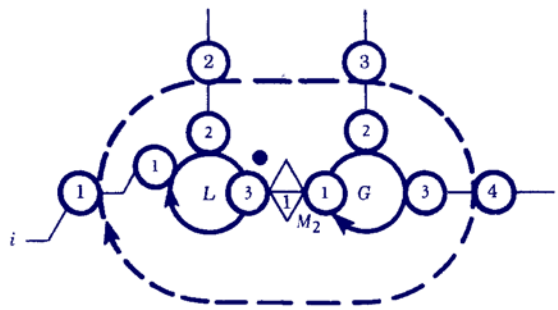


FIG. 19.

Conceptually, we may think of the quantifiers Σ ("some," graphically symbolized by the open triangle –◁) and Π ("all," graphically represented by the solid triangle –◀) as doing two things: first of all, they remove those terms quantified from the running; that is, quantified terms are "used up" and no longer function as blanks in the resulting, external sentence. Secondly, quantification causes the terms quantified to lose their particularity. In the transition from sentence (S34) to sentence (S35), above (Fig. 18 to Fig. 19), "k" is no longer some identifiable object, but is reduced to "somebody," "someone," or "something." It is probably best to replace the "k" with some different notation both in our diagrams and in our subscript notation with something like "one (who)". If there are several such quantified elements, we may refer to them as "one who ..., a two who ..." and so on, being completely free to choose arbitrarily which of these is to be called "one," "two" and so on. We may decide, for example, to use the order in which they happen to appear in the sentence representing a particular structure. The use of cardinal numbers here (in relative structures) may help distinguish the internal relation from that of sentences (S26) and (S27), whose ordinal numbering refers to particular, named elements (in non-relative structures).

* If the operation between L_{ijk} and G_{kem} were a sum, we would first obtain the pentada that reads

"__lies in between__and__, or this last gives__ to__" (S36)

By contracting now on the repeated index by means of the quantifier, we obtain

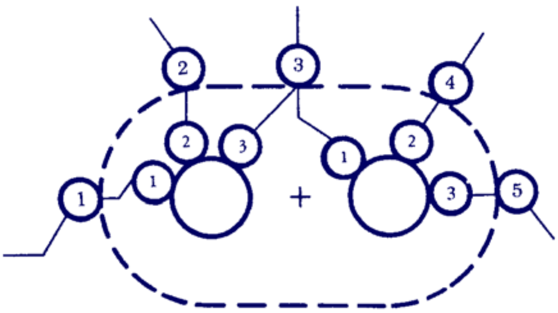
$$N_{ijem} = \prod_{k} L_{ijk} + G_{kem} \qquad [E19]$$

that is,

"take any individual, then either__ lies in between__ and
this individual or this individual gives__ to__" (S37)

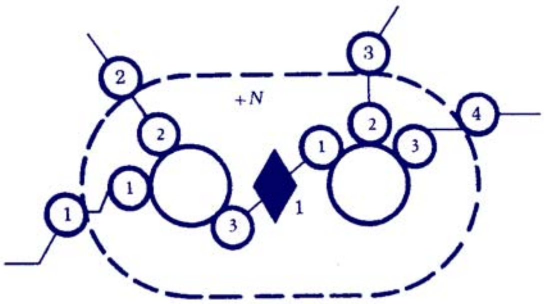
end of *

This is similar to the relative sum of diadas. Sentences number (S3)6 and (S37) have been diagrammed in Figs. 20 and 21, whose differences compared with the Figs. 18 and 19 is first that Fig. 20 and 21 have a + sign instead of a • sign; this change indicates, of course, that the operation carried out is a sum ("or") rather than a product ("and"); second, Fig. 21 has a solid double-triangle where Fig. 19 has an open double-triangle. This, it is clear, indicates that the colligative term in Fig. 19 has been quantified by a Σ ("any" ⊸◁) while that of Fig. 21 has been quantified by a Π ("all" ⊸◀).



S36

FIG. 20.



S37

FIG. 21.

The combination of triadas with colligative terms is amenable (Peirce) to clear graphical representation. We have already seen how colligative terms are shown as "double triangles" in Figs. 19 and 21. We note, however, that the Figs. 18, 19, 20, and 21 are not "well-formed" according to our criterion, since the external sequence is not identical with the internal sequence, and interval connections are not restricted to arms of the same order. Thus, we have used these illustrations only in order to describe what is meant by colligative terms. A well-formed combination of n-adas can only result in another structure with n or *fewer* terms, for no n+1st "arm" can be related to an n+1st arm inside the structure. Figure 17 is a well-formed structure, for example, in which a triada whose third arm is Σ-quantified (i.e., "some"), for the first arm is the first arm and the second arm is the second arm in either the external or the internal structure.

Two other well-formed examples of Σ-quantified triadas resulting in dyadas are shown in Figs. 22 and 23.

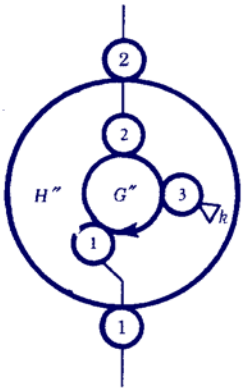


FIG. 22.

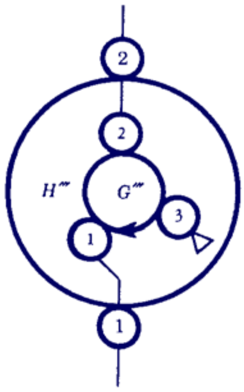


FIG. 23.

G" = " __is given by__to someone" (S38)

where $H'' = \overset{\circ\!\rightarrow}{G''}$ [E20]

and

H" = "__is given by__" (S39)

G"' = "to__is given__by someone, (S40)

where $H''' = \overset{\circ\!\rightarrow}{G'''}$ [E21]

H"' = "to__is given__" (S41)

Note that

$H' \neq H'' \neq H'''$ [E22]

and that

$G' \neq G'' \neq G'''$ [E23]

* For convenience, we shall define closed relative products and sums among triadas in which the contraction or generalization by the quantifiers is realized upon repeated indexes, and in which each repeated index repeats only once. This permits the use of the above-mentioned type of graph as a means for visualizing the relative operations, and, at the same time, provides us with another tool to prove theorems. It turns out thay many of the combinations of open operations which finally result in triadas are particular cases of closed products and sums defined with those rules. Briefly, the rules for forming relative operations of triadas, which permit the use of the above-mentioned graphs, may be stated as follows.

(i) Each repeated index repeats only once.
(ii) Quantifiers act on repeated indexes.

It follows from the graphs that at least three triadas are necessary to verify a closed operation. There are three different ways in which the triadas could be connected (see Figs. 24, 25, and 26). These lead to the relative products and sums that are defined below.

end of *

Relative Products

Δ **Product** of three triadas A, B, and C is the triada

$$Q_{ijk} = \Delta(ABC) = \sum_{123} A_{i23} \cdot B_{1j3} \cdot C_{12k} \qquad \text{[E24]}$$

or, if A = A_{abc}, B = B_{def}, and C = C_{ghl}, then a = i, b = 2, c = 3, d = 1, e = j, f = 3, g = 1, h = 2, l = k, where i, j, and k represent the "free" terms, and 1, 2, and 3 represent the "bound" terms. We should note that the sequence of *numbers* is arbitrary; here w simply taken the order in which they happen to come. The diagram for this pro, shown in Fig. 24, which makes the relationship among triadas visible.

>– **Product** of the triadas A, B, and C is the triada

$$R_{ijk} = \succ\!\!-(ABC) = \sum_{123} A_{ij3} \cdot B_{123} \cdot C_{12k} \qquad \text{[E25]}$$

or, if A , B , and C are as above, then a = i, b = j, c = 3, d = 1, e = 2, f = 3, g = 1, *l* = k.

(This product is diagrammed in Fig. 25.)

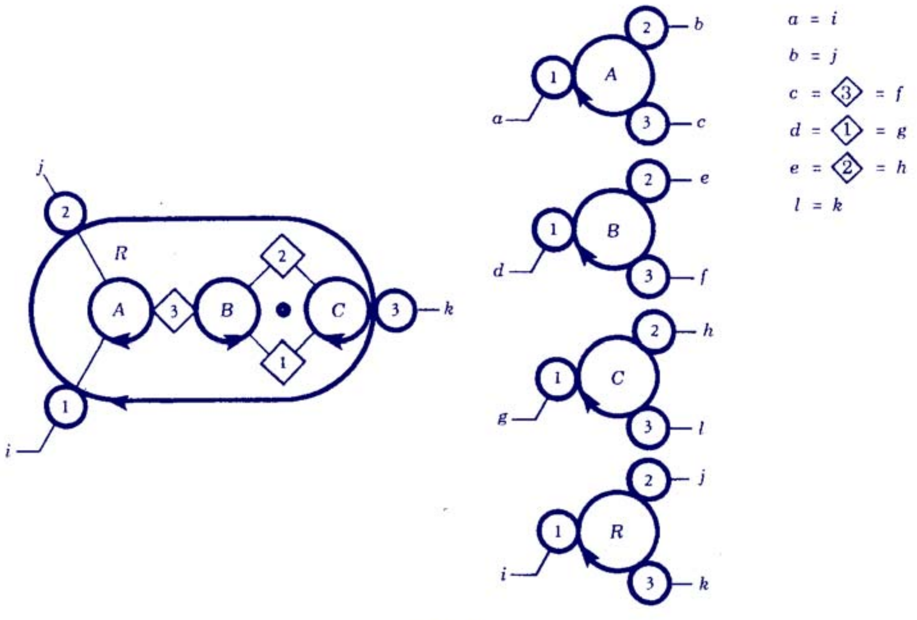


FIG. 25.

(For convenience, we number the colligative terms according to the order of the arms they share here.)

–< **Product** of the triadas A, B, and C is the triada

$$S_{ijk} = -\!\!\prec(ABC) = \sum_{123} A_{i23} \cdot B_{123} \cdot C_{1jk} \qquad \text{[E26]}$$

or, if A , B , and C are as above, then a = i, b = 2, c = 3, d = 1, e = 2, f = 3, g = 1, h = j, *l* = k. (See Fig. 26 for a diagram of this product.)

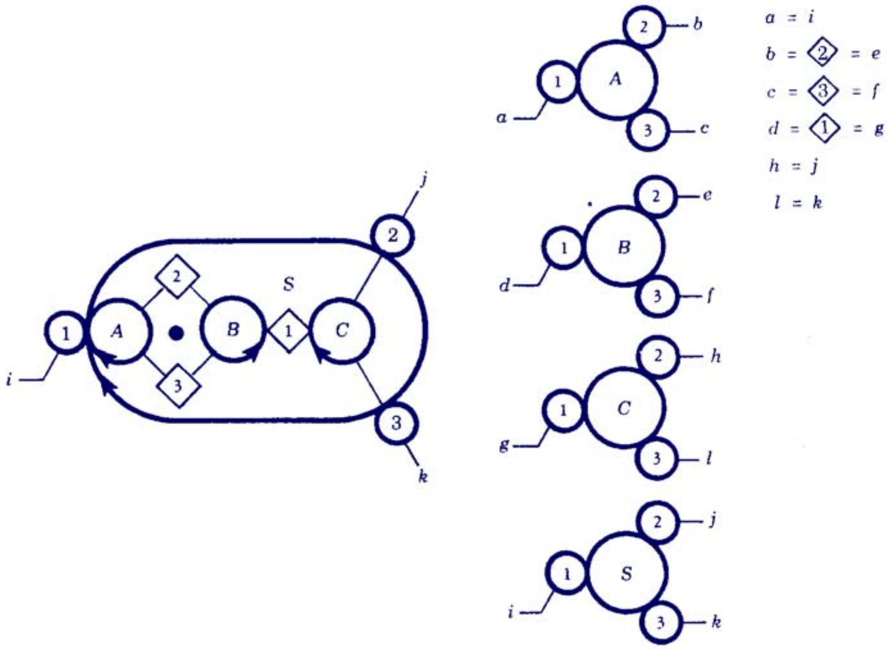


FIG. 26.

In terms of shifts, we might note that B in either Fig. 25 or Fig. 26 only seems free to rotate and reflect, for it happens to have none of its "arms" passing through the "envelope" of the resulting triad. Such is certainly not the case in Fig. 24, however, where B seems free to be reflected only but not rotated. But we must take into account the internal structures as well, for such manipulation changes which arm is "connected" to which arm. The apparent freedom in B is thus illusory.

In all three Figs. 24, 25, and 26, the internal sequence is the same as the external sequence. That is to say, the first arm (i) of the external figure is first arm of the internal triada, the second arm (j) is the second arm internally and externally, and the third arm(h)is the third arm internally and externally. Thus,

$$\overrightarrow{Q}^{\circ} = \underset{\bullet}{\Delta}(ABC) \qquad [E27]$$

$$\overrightarrow{R}^{\circ} = \underset{\bullet}{\succ\!-}(ABC) \qquad [E28]$$

and

$$\overrightarrow{S}^{\circ} = \underset{\bullet}{-\!\prec}(ABC) \qquad [E29]$$

Furthermore, all internal connections are between arms of the same order. These constructions are therefore "well-formed."

For example, if

A = "gives__to__" (S42)

B = "__leaves__for__" (S43)

and C = "__tells on__to__" (S44)

then Δ(ABC) is the triada

"__gives 2 to 3, and 1 leaves__for 3, and 1 tells on 2 to__" (S45)

or

"there are three individuals such that__ gives the second to the third,

and the first leaves__ for the third, and the first tells on the second to__"
(S46)

Relative Sums*

Δ **Sum** of three triadas A, B, and C is the triada

$$Q_{ijk} = \underset{+}{\Delta}(ABC) = \prod_{123} A_{i23} + B_{1j3} + C_{12k} \qquad [E30]$$

end of *

which is shown graphically in Fig. 27. It should be obvious that the sum of three triadas is exactly the same as the product, except that the quantifier is universal instead of existential (Π instead of Δ), and the sum sign (+)is used instead of the product sign (.)

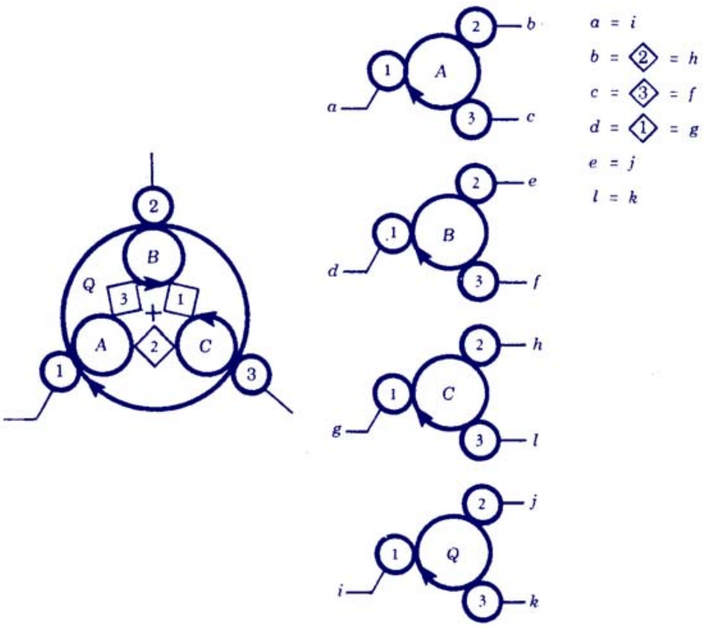


FIG. 27.

>– **Sum** of the triadas A , B , and C is the triada

$$R_{ijk} = \underset{+}{\succ\!\!-}(ABC) = \prod_{123} A_{ij3} \bullet B_{123} \bullet C_{12k} \qquad [E31]$$

which is graphically displayed in Fig. 28.

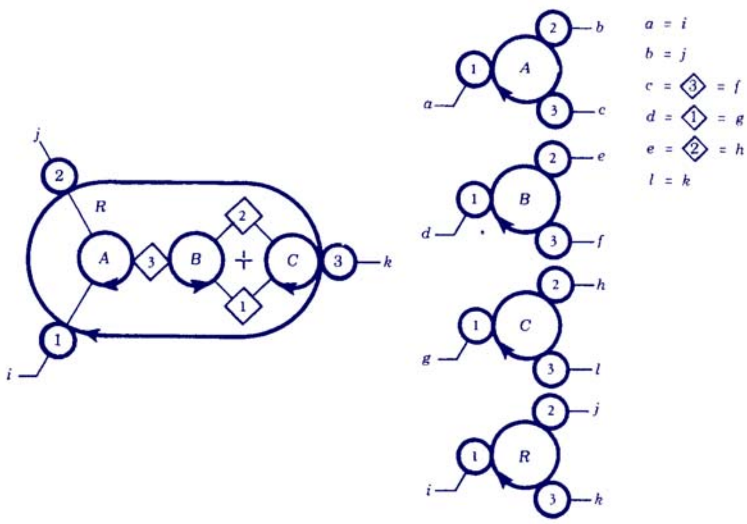


FIG. 28.

Sum of the triadas A , B ,and C is the triada

$$R_{ijk} = \underset{+}{\prec}(ABC) = \prod_{123} A_{i23} \bullet B_{123} \bullet C_{1jk} \quad [E32]$$

which is shown in Fig. 29.

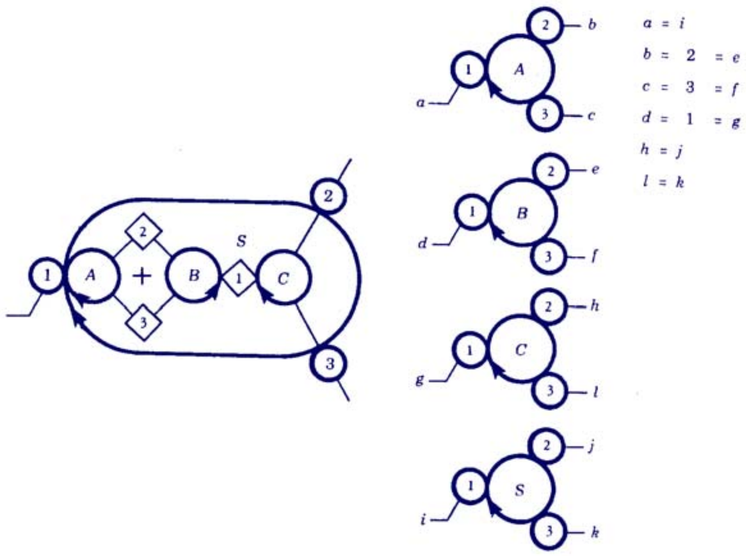


FIG. 29.

For example, using A , B , and C as in (S42, S43, and S44, above), then $\underset{+}{\Delta}(ABC)$ reads

> "Take any three individuals; then, either i gives 2 to 3, or 1 leaves j for 3, or the 1 tells on 2 to k " (S47)

Before we leave the subject of notation of relative sums and products, we might observe that again, the significant features are the origin and the order of the elements, whether these be internal or external. Thus, both diagrams in Fig. 30 are equivalent to the one in Fig. 24, and all four diagrams are equivalent in Figs. 31 as are the four in Fig. 32.

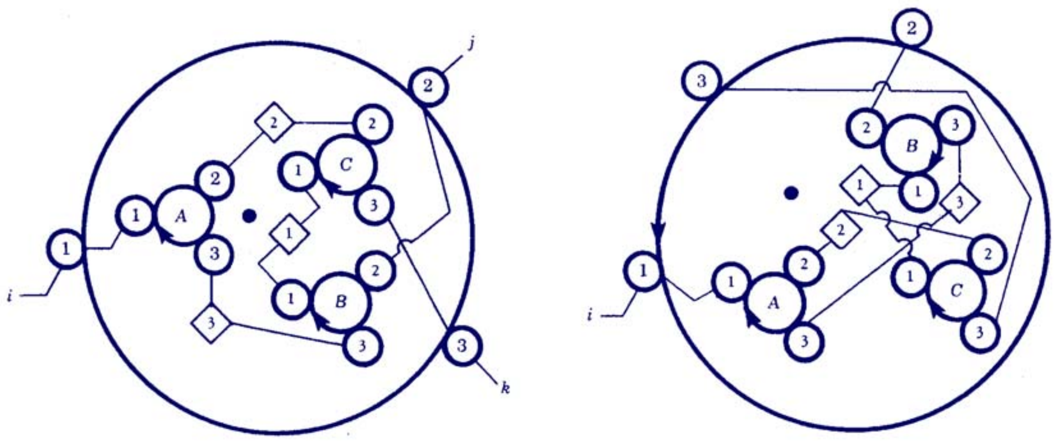


FIG. 30.

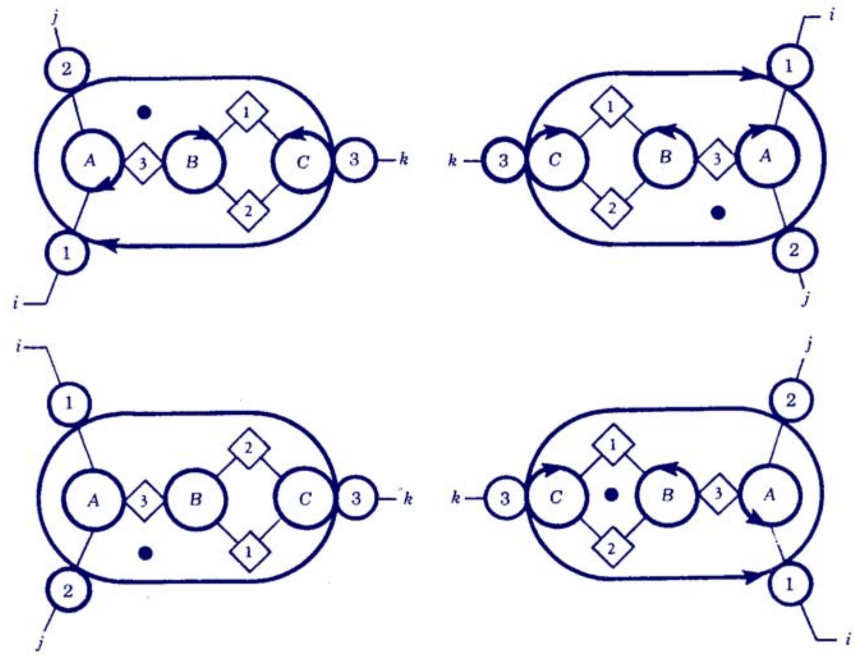


FIG. 31.

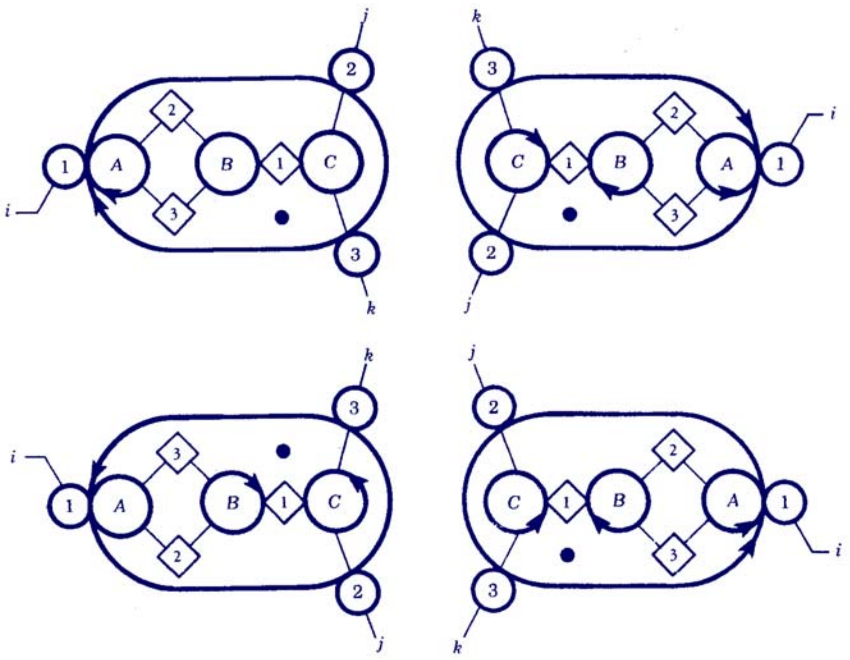


FIG. 32.

Résumé of Closed Operations for Triadas *

Unary
- Rotation, $\overset{\frown}{A}$
- Rotation, $\overset{\smile}{A}$
- Shift, $\overset{\circ\!\rightarrow}{A}$, $\overset{\rightarrow\!\circ}{A}$

Binary
- Nonrelative Product, $A \bullet B$
- Nonrelative Sum, $A + B$,

Triadic
- Relative Products
  - $\underset{\cdot}{\Delta}$(ABC)
  - $\succ\!\cdot$ (ABC)
  - $\cdot\!\prec$ (ABC)
- Relative Sums
  - $\underset{+}{\Delta}$(ABC)
  - $\underset{+}{\succ\!-}$(ABC)
  - $\underset{+}{-\!\prec}$ (ABC)

Immediate Theorems. By combining closed operations among triadas, we can prove the set of equalities, or theorems, that follow.

end of *

Unary Theorems.

We have already noted that $\overset{\frown}{G}_{ijk} = G_{kij}$ [El], that $\overset{\smile}{G}_{ijk} = G_{kji}$ [E3]
and that

$$\overset{\smile}{\overset{\smile}{G}} = G \text{ and that } \overset{\frown}{\overset{\frown}{\overset{\frown}{G}}} = G \text{ [E6], [E7].}$$

Further application of rotation and reflection to a simple triada results in the rest of the six configurations possible for a simple triada to assume, shown in Table 1.

TABLE 1.

A_{abc} = _a_ gives _b_ to _c_.	$G = G_{ijk}$	(S48)	[E33]
$\overset{\frown}{A}_{abc} = A_{cab}$ = to _c_, _a_ gives _b_.	$G = G_{kij}$	(S49)	[E34]
$\overset{\smile}{A}_{abc} = A_{cba}$ = to _c_ is given _b_ by _a_.	$G = G_{kji}$	(S50)	[E35]
$\overset{\frown\frown}{A}_{abc} = A_{bca}$ = _b_ is given to _c_ by _a_.	$G = G_{jki}$	(S51)	[E36]
$\overset{\times}{A}_{abc} = A_{bac}$ = _b_ is given by _a_ to _c_.	$G = G_{jik}$	(S52)	[E37]
$\overset{\circ}{A}_{abc} = A_{acb}$ = _a_ gives to _c_, _b_.	$G = G_{ikj}$	(S53)	[E38]

Figure 33 shows the graphic equivalent of Table 1.

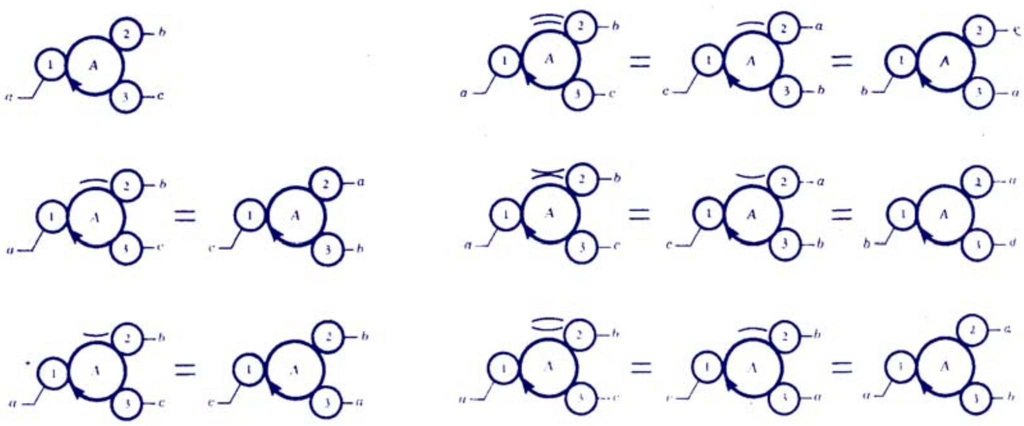


FIG. 33.

Similarly, it is simple to show that

$$\overset{\smile}{\overset{\frown}{\overset{\frown}{A}}} = \overset{\frown}{\overset{\smile}{A}} \text{, (called a ´flop´) or that} \qquad \text{[E39]}$$

$$\overset{\frown}{\overset{\smile}{\overset{\frown}{A}}} = \overset{\smile}{A} \text{ , } \overset{\frown}{\overset{\frown}{\overset{\smile}{A}}} = \overset{\smile}{\overset{\frown}{A}} \text{ and } \overset{\smile}{\overset{\smile}{\overset{\frown}{A}}} = \overset{\frown}{\overset{\frown}{A}} \qquad \text{[E40]}$$

Graphical proofs are indicated in Fig. 34. It should be clear that the "stacked" diacritical marks are to be applied from the inside out in the diagrams, just as they are the algebraic notation.

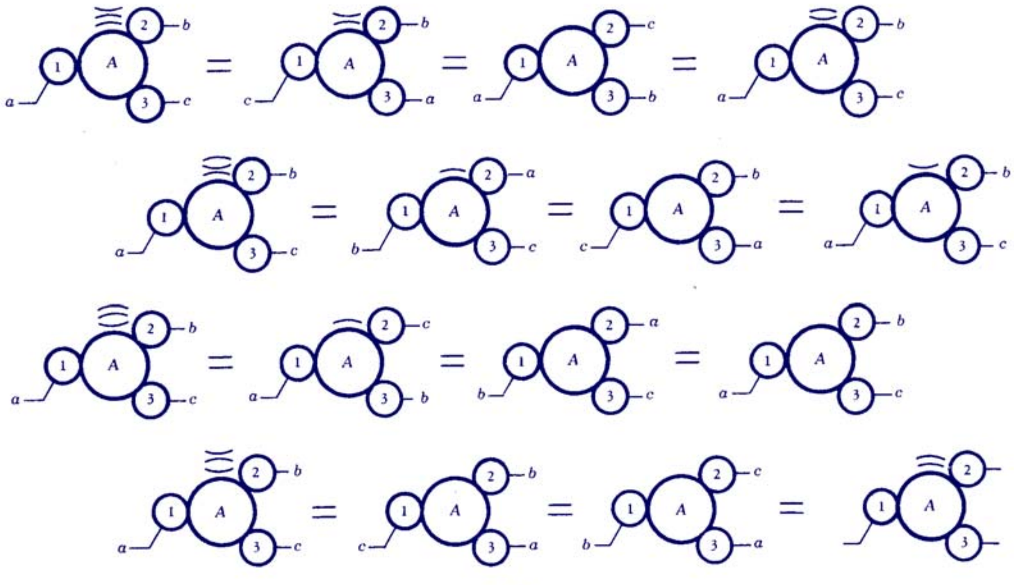


FIG. 34.

Binary, or Nonrelative, Theorems. * Let P_{ijk} be the triada that results from the nonrelative product of A_{ijk} and B_{ijk}. That is,

$$P_{ijk} = A_{ijk} \bullet B_{ijk} \qquad \text{[E41]}$$

Rotation of P_{ijk} gives:

$$\widehat{P_{ijk}} = P_{kij} = A_{kij} \bullet B_{kij} = \widehat{A_{ijk}} \bullet \widehat{B_{ijk}} \qquad \text{[E42]}$$

that is,

$$\widehat{P_{ijk}} = \widehat{A_{ijk}} \bullet \widehat{B_{ijk}} \qquad \text{[E43]}$$

Since the subscripts appear now in the same order on both sides of the last equation, we may now delete them to obtain

$$\widehat{A \bullet B} = \widehat{A} \bullet \widehat{B} \qquad \text{[E44]}$$

end of *

This proof is sketched graphically in Fig. 35.

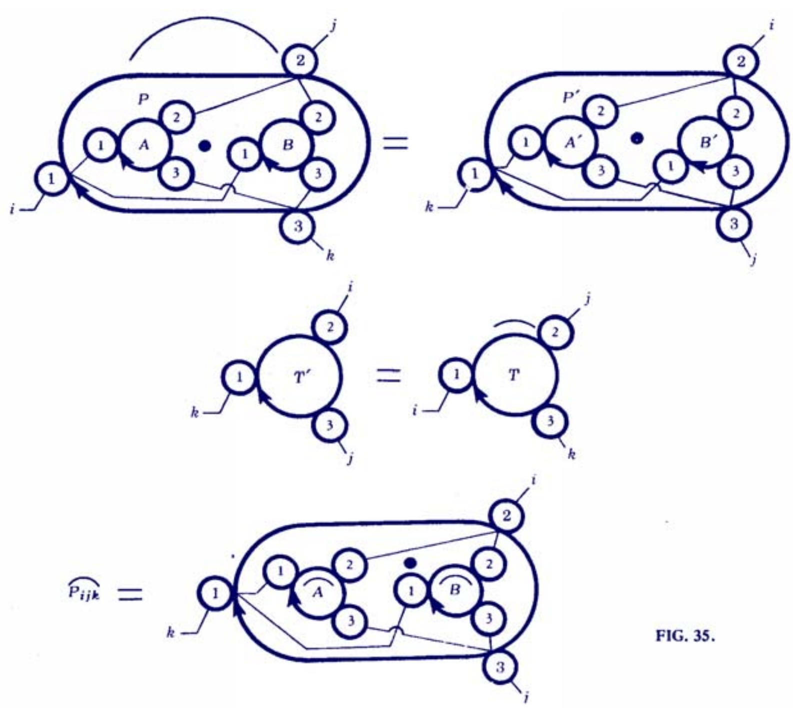


FIG. 35.

Similarly, we can prove that

$$\widehat{A+B}=\widehat{A}+\widehat{B} \qquad [E45]$$

The graphic proof is obtained by replacing every "•" with "+" in Fig. 35. By the same method, we can prove that

$$\breve{A \bullet B}=\breve{A} \bullet \breve{B} \qquad [E46]$$

Its graphical solution is shown in Fig. 36.

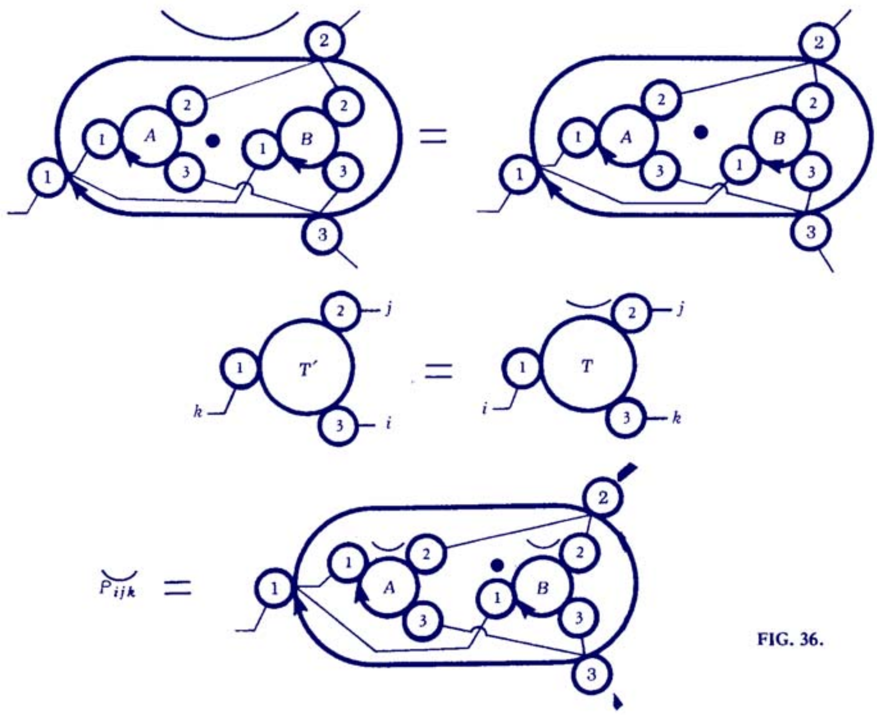


FIG. 36.

Again, replacing "•" by "+" in Fig. 36, we obtain the graphical proof for

$$\overset{\frown}{A + B} = \overset{\smile}{A} + \overset{\smile}{B} \qquad [E47]$$

To assure ourselves of equality in these binary theorems, let us check on whether meaning of the first of our theorems is synonymous throughout the proof.

Let A_{abc} = "a gives b to c" and (S54)

B_{abc} = "a leaves b for c" (S55)

$P_{abc} = A_{abc} \bullet B_{abc}$ = "a gives b to c and the first leaves the second for the third" (S56)

$\overset{\frown}{P_{abc}} = P_{cab}$ = "to c, a gives b and for the first, the second leaves the third" (S57)

$\overset{\frown}{A_{abc}} = A_{cab}$ = "to c, a gives b" (S58)

$\overset{\frown}{B_{abc}} = B_{cab}$ = "for c, a leaves b" (S59)

$\overset{\frown}{A_{abc} \bullet B_{abc}} = A_{cab} \bullet B_{cab}$ = "to c, a gives b ; and for c , a leaves b" (S60)

We thus observe that synonymy is, indeed, maintained.

In Parts 1 and 2 of this paper, I have revised the McCulloch-Moreno-Diaz paper up through unary and binary theorems. So closely has that paper been followed that about a quarter of this text is nearly identical with it. Very little similar text remains to be seen, however, for the divergencies have begun to overwhelm the similarities. Triadic theorems and a section on computer-aided triadic logic development will be taken up in Part 3, which is to appear in a subsequent issue of this journal.

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